COMPLEMENTARY ALGEBRA AND ITS APPLICATIONS ON THE REPRESENTATION OF COMPLEX NUMBERS BY COLORS

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In many applications negative value and positive value of a variable are of different nature (biochemical mediators and hormones, neural nets, colors in human perception and others), so the sum of equal amounts of complementary substances is invariant in respect to the substances added. For instance, addition of any two complementary colours on colour circle gives white colour [Ref 1-6]. It is natural to interpret the sum of complementary substances as zero. In the present contribution we demonstrate how one can construct algebra (for instance, real, complex and quaternion numbers) and other algebraic structures based on regular representation of a group and projective geometry on n-dimensional cones with n variables with non-negative values. In particular, all points of a cone, which are projected into zero, are interpreted as having zero value. Any two points on cone, which sum is projected to zero, we will call complementary. The representation permits to use powerful methods of the theory of analytical functions, differential geometry and topology in new areas.

Let G be a finite group and R[G] be the ring of a regular representation of G over real numbers. Let $R^{+1}[G]$ be a positive octant in R[G]. It consists of linear combinations $\Sigma a_i g_i$, $a_i \subset R$, $a_i \geq 0$, $g_I \subset G$. It is clear that $R^{+1}[G]$ is invariant under summation and multiplication. R[G] contains R as a subring of constant functions on G. In fact, R[G] decomposes into a direct sum of the matrix rings of three types : M(n,R), M(l,C), M(s,H) where r, C, H are the rings of complex numbers and quaternions respectively. and M(n,R) is the ring of n times n - matrices over R and similarly the other ones are the rings of matrices over C,H-respectively.

Consider the projection R[G] to R[G]/R where R is a G-invariant subspace generated by Σg_i, $g_i \subset G$ Then the map $R^+[G]$ to R[G]/R is surjective. Indeed for any $x= \Sigma a_i g_i$ we can find m > 0 such that $a_i +m >0$ for any i. This gives a representation of any element x in R[G]/R as the image of some element $x+me \subset R^{+}[G]$.

Corollary. Any algebra in the above decomposition of R[G] can be realized by positive elements with linear equivalence. In particular if $G=Z_3$ then R[G]/R = C and we obtain realization of complex numbers by triples of positive real numbers. Similarly if $G =Q_8$ is a quaternion group generated by i, j, k, ij = k = -ji, jk =i= -kj, ki =j =-ik ,$i^2=j^2=k^2= -1$. In this case have a natural decomposition $R[Q_8] = H + R +\Sigma R_i$ were we make a summation over 3 nontrivial Z_2-characters of Q_8. The elements of Q_8 form a natural system of 8 vectors in H so that any element can be presented uniquely as a sum of four linearly independent elements from Q_8 with nonnegtaive coefficients. Similar picture holds for the representations R[G/H]. The group acts by permutations on the set G/H and R[G/H] always contains G-invariant subspace R. The map of $R^{\{+\}}[G/H]$ to R[G/H]/ is always surjective.

Example Any system of basic roots of semisimple Lie algebra provides with a similar property for the corresponding real Cartan sublagebra k. If e_i are such roots then any element in k has a unique representation as a sum of e_i with nonnegative coefficients. Note that the first example related to Z-3 corresponds to the root system A_2.

It sounds natural to give these matrices a special name. Because the matrices on cones in general correspond to idea of complementarity (nucleotides, colours, etc.) we give them the name **complementary matrices**. For instance, in case of 3D-cone-representaion of algebra of complex numbers one has one complementary projection, but in case of 4D-cone-representation of algebra of complex numbers one has two complementary projections, one for real and one for imaginary numbers. One can note that formalism is using well known fact that the sum of all matrices of a regular representation of any finite group is the matrix with all elements equal to one (and its complementary projection is zero) [Ref 7].

The explicit form of complementary matrices generated by the rotation group C_4 is the following:

$$\begin{pmatrix} a & c & d & b \\ b & a & c & d \\ d & b & a & c \\ c & d & b & a \end{pmatrix} = \begin{cases} \begin{pmatrix} a-d & 0 & 0 & b-c \\ b-c & a-d & 0 & 0 \\ 0 & b-c & a-d & 0 \\ 0 & 0 & b-c & a-d \end{pmatrix} & \text{if } a \geq d, b \geq c \\ \begin{pmatrix} a-d & c-b & 0 & 0 \\ 0 & a-d & c-b & 0 \\ 0 & 0 & a-d & c-b \\ c-b & 0 & 0 & a-d \end{pmatrix} & \text{if } a \geq d,\ c > b \\ \begin{pmatrix} 0 & 0 & d-a & b-c \\ b-c & 0 & 0 & d-a \\ d-a & b-c & 0 & 0 \\ 0 & d-a & b-c & 0 \end{pmatrix} & \text{if } d > a, b \geq c \\ \begin{pmatrix} 0 & c-b & d-a & 0 \\ 0 & 0 & c-b & d-a \\ d-a & 0 & 0 & c-b \\ c-b & d-a & 0 & 0 \end{pmatrix} & \text{if } c > b, d > a \end{cases} \quad (1)$$

In particular operational **1, î, -1, - î** in this 4d complementary representation of the algebra of complex numbers are the following matrices:

$$\hat{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$-\hat{1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\hat{i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$-\hat{i} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(2)

VECTOR GROUP PRUDUCT. Along with the matrix complementary formalism, one can construct the vector representation of complementary objects [Ref 8]. In order to do it, one should determine the **vector group product** [Ref. 8]. For the group C_4 (which we consider in the present contribution in reference to complex numbers) vector group product is determined as follows:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{pmatrix} ae + bh + cg + df \\ be + ch + dg + af \\ ce + dh + ag + bf \\ de + ah + bg + cf \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = xc_1 + yc_i + zc_{-1} = wc_{-i}$$

(4)

VECTOR COMPLEMENTARY PRODUCT As it has been proven above, in order to get algebra, generated by some group, one much perform complementary operation as well. The vector **complementary product** is a combination of two operations: 1) vector group product and 2) complementary projection. For the group C_4 the vector group product has the following form:

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \otimes \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \perp \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} * \begin{pmatrix} e \\ f \\ g \\ h \end{pmatrix} = \begin{cases} (x-z)c_1 + (y-w)c_i & \text{if} \quad \mathrm{x} \geq \mathrm{z} \quad \text{and} \quad \mathrm{y} \geq \mathrm{w} \\ (x-z)c_1 + (w-y)c_{-i} & \text{if} \quad \mathrm{x} \geq \mathrm{z} \quad \text{and} \quad \mathrm{w} > \mathrm{y} \\ (z-x)c_{-1} + (y-w)c_i & \text{if} \quad \mathrm{z} \geq \mathrm{x} \quad \text{and} \quad \mathrm{y} \geq \mathrm{w} \\ (z-x)_{-1} + (w-y)c_{-i} & \text{if} \quad \mathrm{z} > \mathrm{x} \quad \text{and} \quad \mathrm{w} > \mathrm{y} \end{cases} \tag{5}$$

Addition of complementary vectors is standard, e.g. it is determined as some of two vectors in linear algebra and does not need any additional operation or defition. In this formalism, **complementary projection** of the sum of two complementary vectors equals zero. Below are presented two pairs of complementary vectors:

$$\begin{pmatrix} a \\ 0 \\ c \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ a \\ 0 \\ b \end{pmatrix} \quad , \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ and } \begin{pmatrix} b \\ a \\ d \\ c \end{pmatrix} \tag{6}$$

One can show that both matrix and vector representation of **complementors** generate the same algebra. The simplest case of complementors is generated by the group C_2. It generates the algebra of real numbers. The group D_2 generates 4-dimensional representation of so called *hexagonal numbers* (which to not always have inverse elements). A very special case of complementors (8x8 matrix- or 8 dimensional vector-algebra), is generated by the regular representation of the quaternions group are spinors [Ref 9, 10]. In applications, this representation might be useful, in particular, for providing rotations in 3D space [Ref. 11].

THREE DIMENSIONAL COMPLEMENTARY RERPESENTATION OF COMPLEX NUMBERS. One can represent the algebra of complex number not only by 4-dimensional complementors, but by 3-dimensional complementors as well {Ref 9]. 3D-representation is generated by the group of rotations , e.g. planar rotations by the angles 0, $2\pi/4$ and $4\pi/3$ around the fixed axes. Complementary matrices in 3D representation are as follows:

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} = \begin{cases} \begin{pmatrix} a-c & b-c & 0 \\ 0 & a-c & b-c \\ b-c & 0 & a-c \end{pmatrix} \text{if a} \geq \text{c}, \text{b} \geq \text{c} \\ \\ \begin{pmatrix} a-b & 0 & c-b \\ c-b & a-b & 0 \\ 0 & c-b & a-b \end{pmatrix} \text{if a} \geq \text{b}, \text{c} > \text{b} \\ \\ \begin{pmatrix} 0 & b-a & c-a \\ c-a & 0 & b-a \\ ba & c-a & 0 \end{pmatrix} \text{if b} > \text{a}, \text{c} > \text{a} \end{cases} \tag{6}$$

In this representation, complex numbers are generated by four semi-axes rather than my two axes (real and imaginary). The unit matrices of four semi-axes, generating the algebra of complex numbers, are the following ones:

$$\hat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad -\hat{1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\hat{i} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \qquad -\hat{i} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \tag{7}$$

Lorentz transformation and complementary transformation. Let consider 4-dimensional complementors generated by the group C_4. The rotation of a coordinate axes by the angle Ψ rotates complementary axes on the same angle in the opposite direction.

$$a' = \frac{a - b\,\mathrm{tg}\Psi}{1 + \mathrm{tg}\ \Psi} \qquad \mathrm{b}' = \frac{\mathrm{b - a\ tg}\Psi}{1 + \mathrm{tg}\ \Psi} \tag{8}$$

The angle by which coordinates are rotated as a result of two sequential rotations of complementary axes by angles 1 and 2 coincides with corresponding transformation of Lorentz group [Ref 12]:

$$tg(\Psi) = \frac{tg(\Psi_1) + tg(\Psi_2)}{1 + tg(\Psi_1)\, tg(\Psi_2)}$$

(9)

but transformation of coordinates themselves after rotation by the angle gives a formula (9) different from that in the theory of relativity of Einstein [Ref. 12].

Discussion.

1. Applications of projective-algebraic approach to neural nets, both in technology and in vivo, seems to be natural. In particular, using this approach, one can construct algebraic structures of visual images. At present, such a prospective looks very promising. Future research will show how effective complementary algebra will be in practical applications in television and internet.

2. The idea that the creation of algebraic structures in the brain is an important part of the processes of thinking and recognition, is commonly accepted. However the idear the complementary algebras can constitute the bases of the neural nets functioning, looks new. It is a common knowledge that biochemical mediators in enzyme and intercellular reactions work on the complementary chemical principle, compensating effects of each other. The complementary algebra permits to translate the intuitive concepts of complementarity on mathematical language.

3. The approach sounds equally effective for color vision analysis. As it is very well known, three basic colours on one hand, correspond to frequences (which are linearly organized), and, on the other hand, consitute colour cirle. Addition of any two colours of equal intensity, occupying positions on opposite sides of the colour circle, make white colour, and any three colours with angle 2¹/3 between them, give white colour too. So colours are added like complex numbers. If in addition, one corresponds to colour saturation the dencity of a function of complex variable, we get one-to-one correspondence between functions of complex variable and colour images. The formation of colour circle in accord with the scheme of Young-Helmholtz (with transfomations on 3-dimensional cone) is given on Fig.1. The formation of colour circle in accord with Heringós scheme (with transfomations on 4-dimensional cone) is given on Fig.2. The future research will show how productive this algebraic-geometrical approach will be in applications to pattern recognition, signal transmission, photography, movie production and other applications.

4. The properties of images in algebraic representation depend on both local and global parameters. The statement that computational programs and even computers based on the complementary algebra will be equally (or even more) effective then traditional ones, based on Boolean algebra, at least in some applications, sounds intriguing.

5. The inverse operation, namely the possibility of the representation of complex number by colors is straightforward. It follows from the formalism, that there is one to one correspondence between 4D complementors generated by the rotation group C and the algebra of complex numbers. As a result this formalism might be used at school and high school programs for complex numbers teaching. It might be equally effective in any research, which involves the analysis of the functions of complex variables.

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Figures

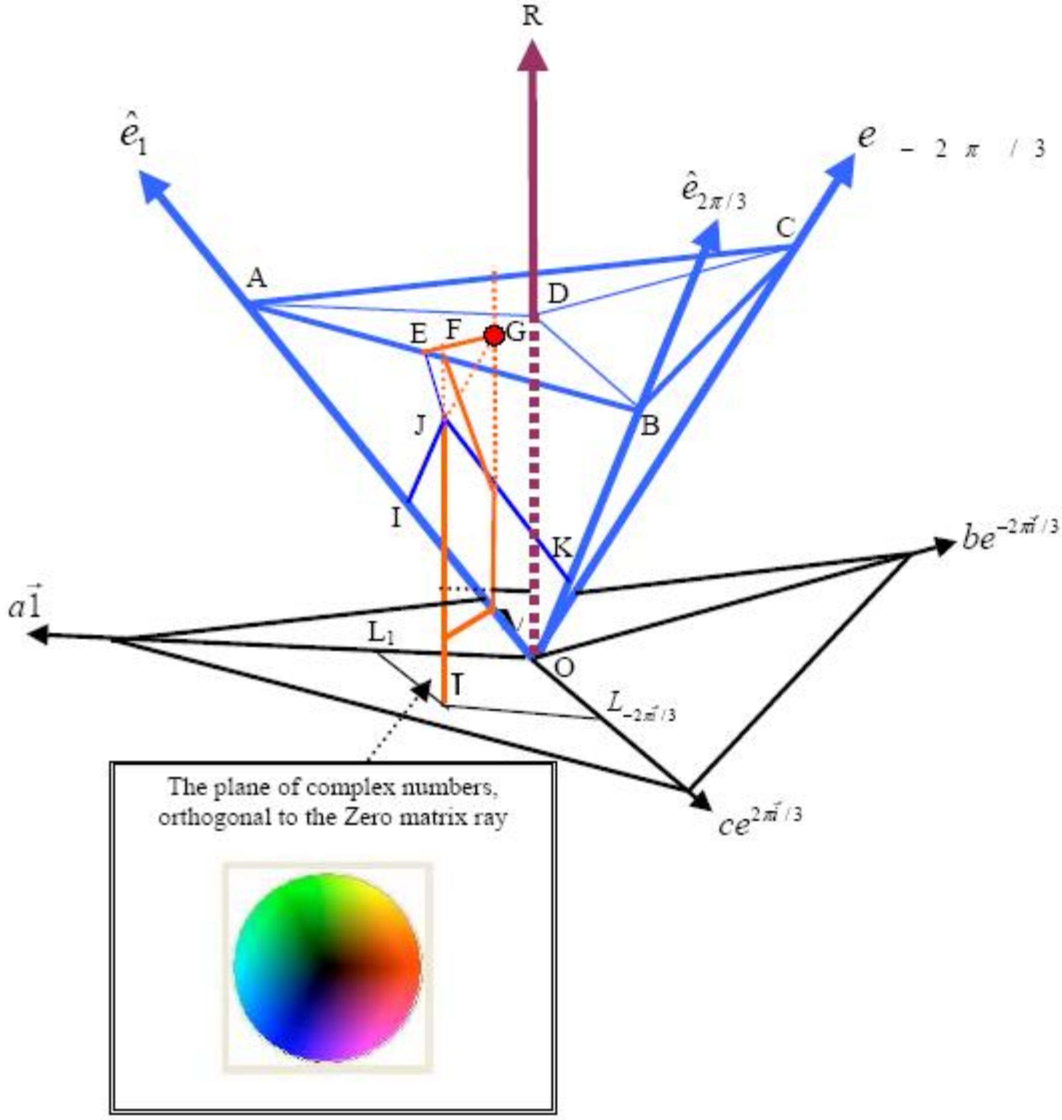


Fig.1 The formation of colour circle in accord with the scheme of Young-Helmholtz.
a. Any three non-negative numbers determine the image F.
b. One projects F parallel to the white ray, i.e. to the ray which forms equal angles with all three basic axes. This projection is cross-section one of three basic quadrants. So one finds the complementor J of the image F.
c. One finds the complementary projection L of complementor J (or of the image F) on the plane, orthoronal to the white ray and passing through the coordinates origin.

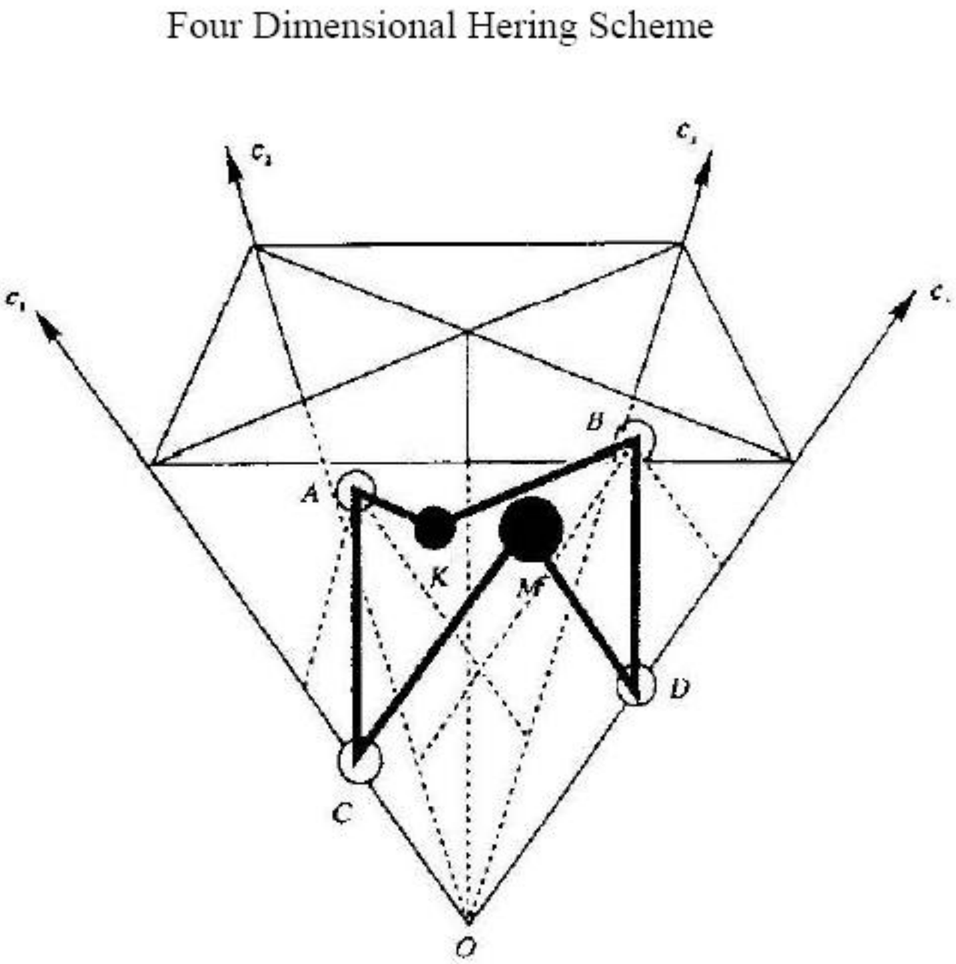
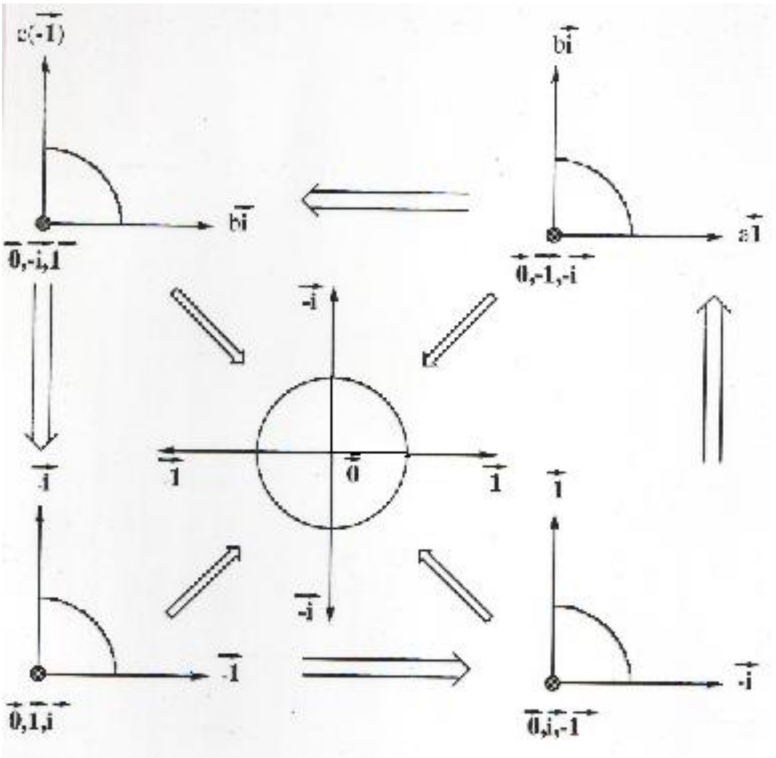


Fig.2 The formation of colour circle in accord with Heringós scheme. 4 basic rays determine 4D-vector (a,b,c,d). There are altogether 6 combinations of pairs of basic rays. They constitute 4 basic quadrants
in 4D-space. If one of them (real) is selected somehow, complementary (imaginary) quadrant is also determined. Remainig 4 quadrants make a circle. This cyclic structure of the space corresponds to the structure
of cyclic product. This complementary scheme generates 6-fold path in the 4-dimensional image space.

a. 4D-image K is projecting onto the real and imaginary basic quadrants. Let these projections are A and B.

b. In real and imaginary quadrants one draws complementary projections. So two complementors C and D have been constructed.

c. One constructs 4-dimensional image M which projections are C and D. Complementor C determines real part of complex number M, as complementor D is its imaginary part.



Fifugre 3. Формирование круга комплексной плоскоси из четырех сегментов, лежащих на ортогональных другу в четырехмерном пространстве квадрантах.

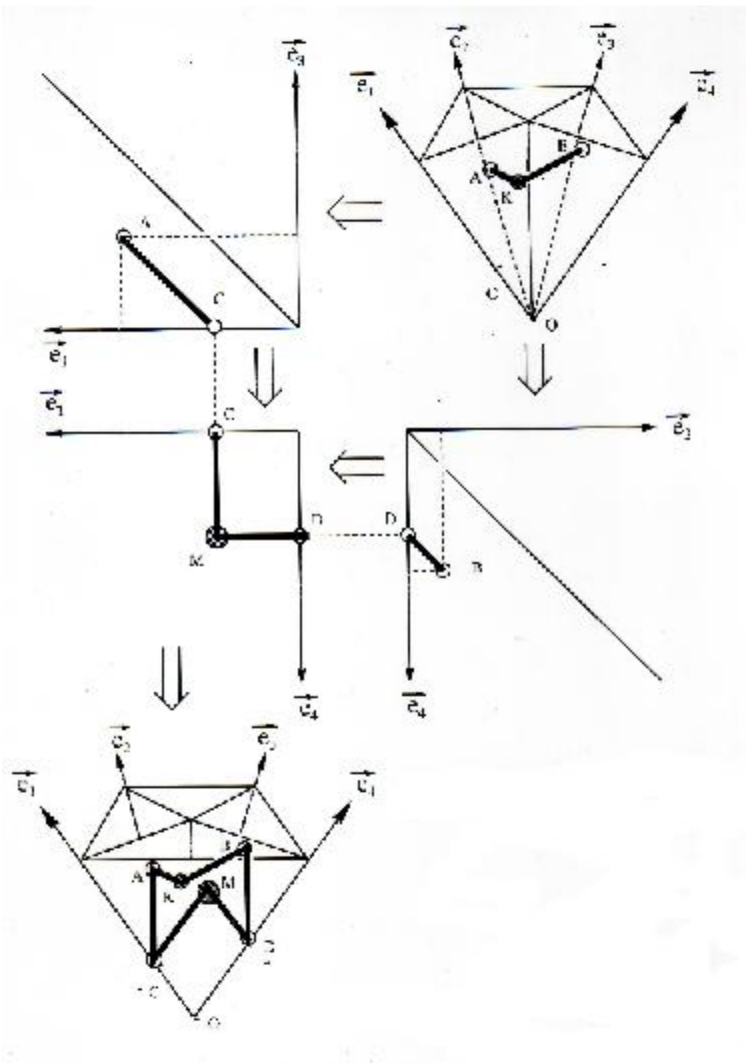


Figure 4. Иллюстрация формирования комплементарной проекции в четырехмерном пространстве.

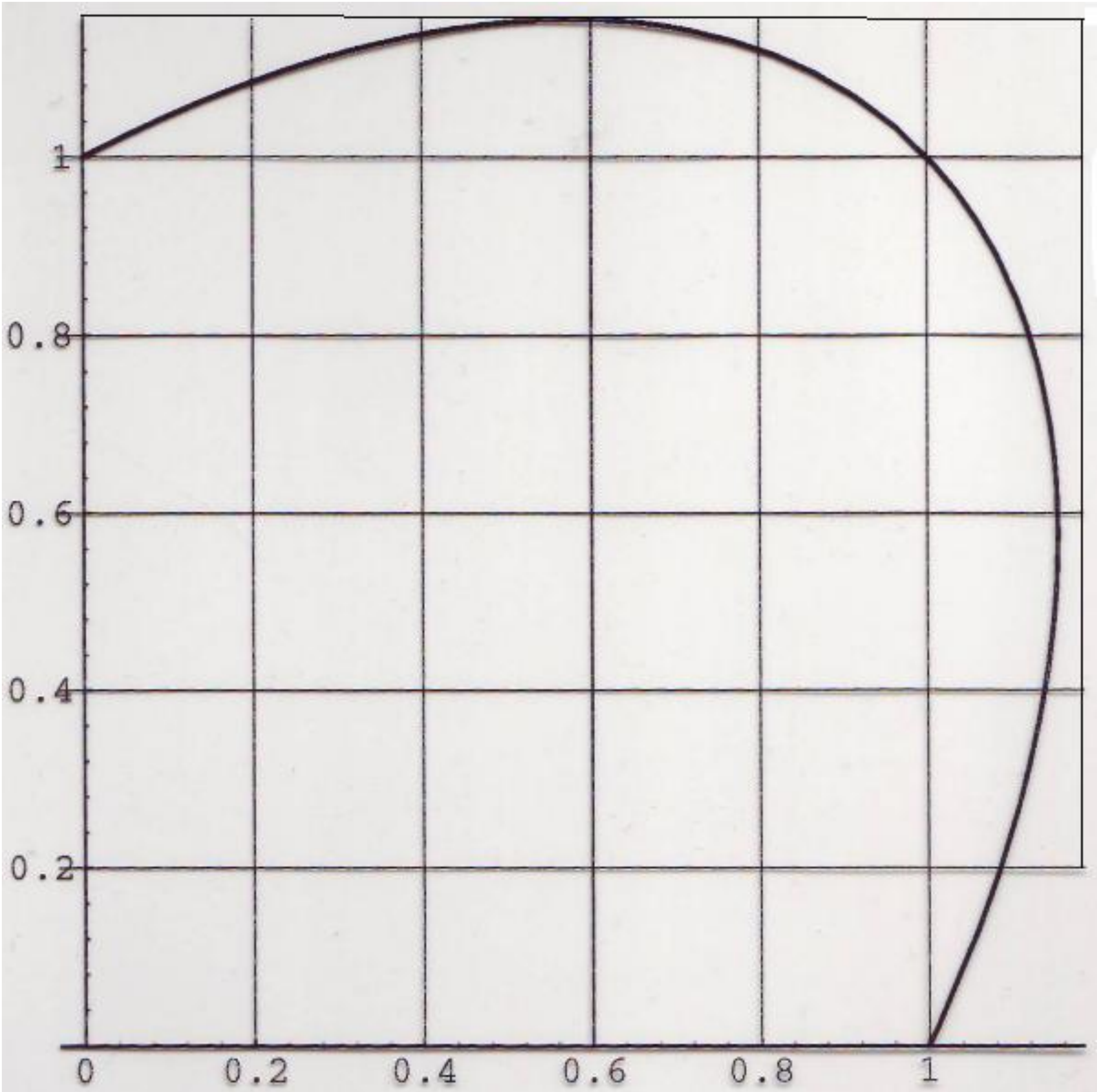


Figure 5. Форма кривой на каждом из четырех квадрантах в четырехмерном пространстве, при проекции на (комплексную) плоскость дающую сегмент с разностью фаз $\Delta\varphi=\pi/2$. В сумме эти четыре кривых в проекции на (комплексную) плоскость дают единичый круг.

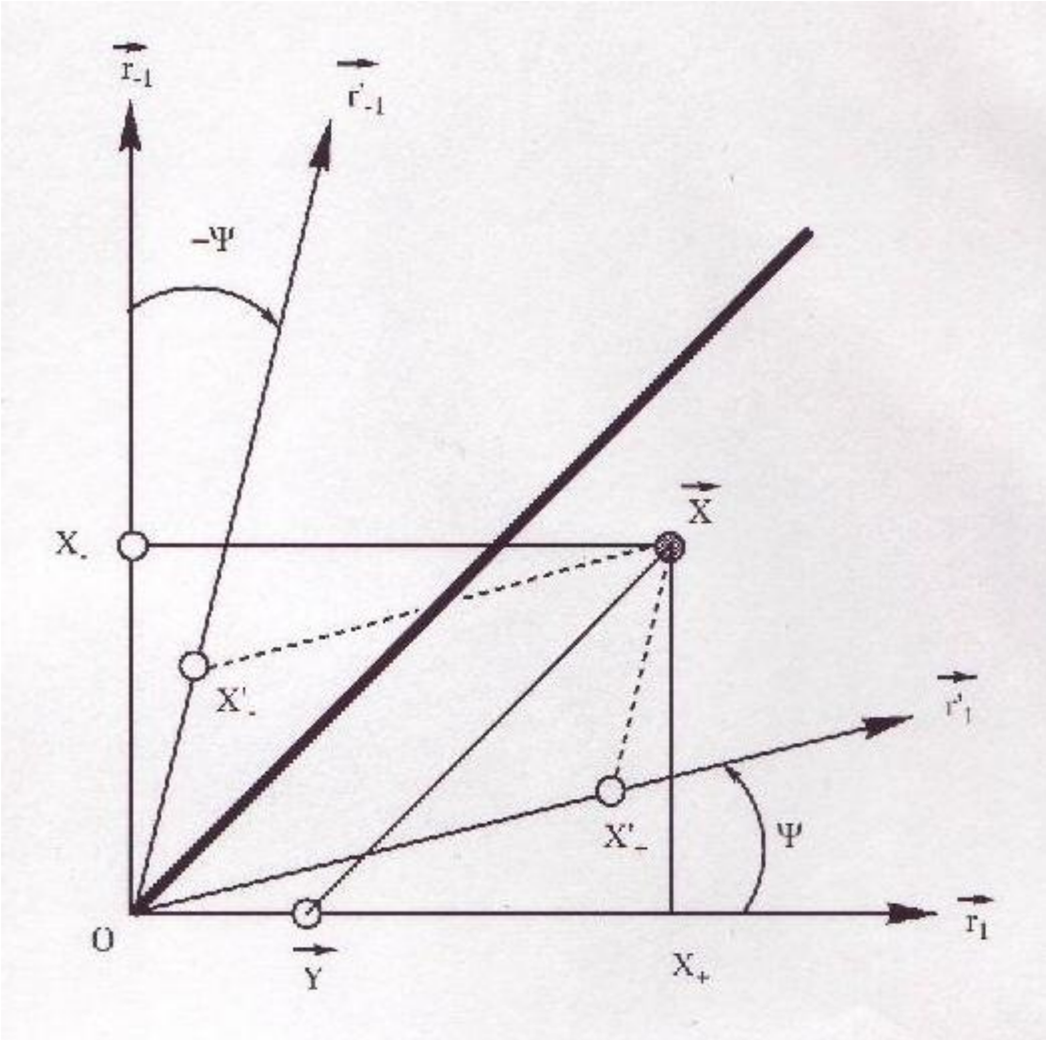


Рис.6. Преобразование углов. При повороте на угол   полуоси квадранта поворачиваются в противоположные стороны на равные углы, как и в преобразовании Лоренца. Координаты точки до поворота обозначены без штрихов, после поворота в новых координатах – с штрихами.

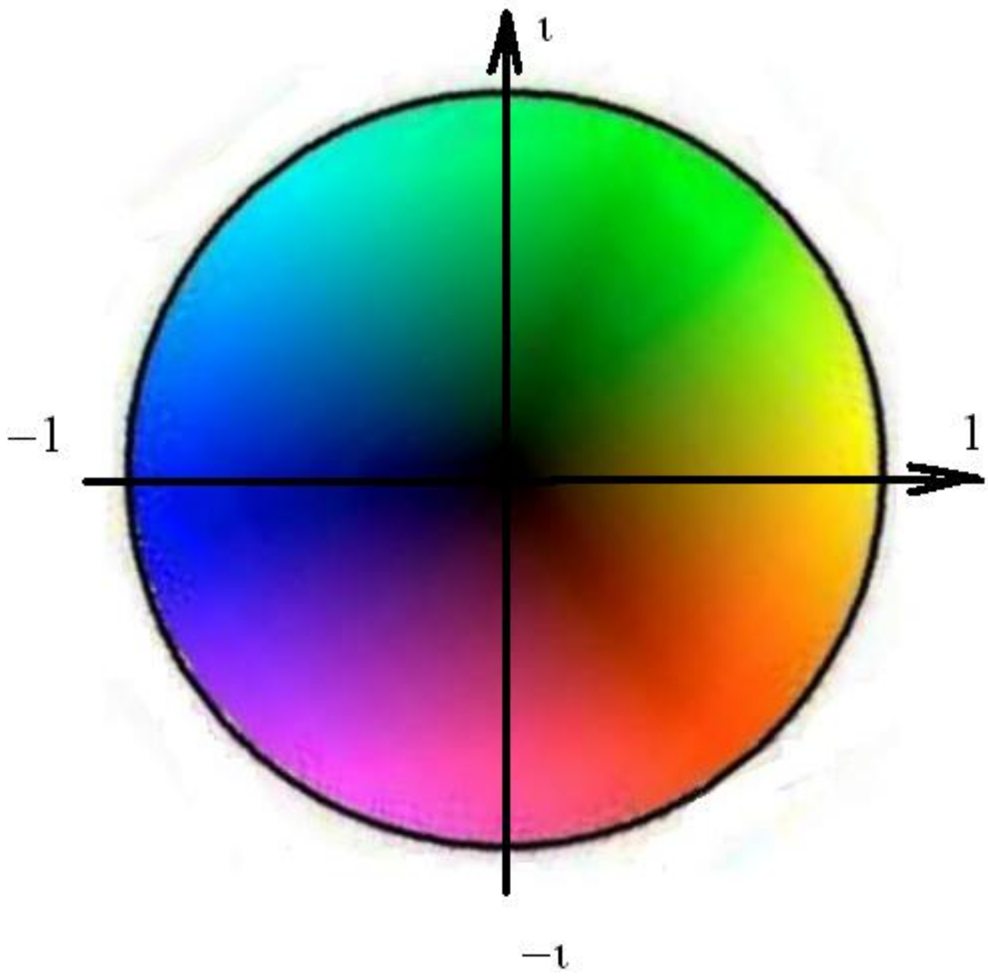


Fig. 7 The color circle which is generated and used in the computer program, which represents complex numbers by colors.