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COHERENCE IN TURBULENCE: NEW PERSPECTIVE

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Abstract

It is claimed that turbulence in fluids is inherently coherent phenomenon. The coherence shows up clearly as strongly correlated helicity fluctuations of opposite sign. The helicity fluctuations have cellular structure forming clusters that are actually observed as vorticity bands and coherent structures in laboratory turbulence, direct numerical simulations and most obviously in atmospheric turbulence. The clusters are named **BCC** - Beltrami Cellular Clusters - because of the observed nearly total alignment of the velocity and vorticity fields in each particular cell, and hence nearly maximal possible helicity in each cell; although when averaged over all the cells the residual mean helicity in general is small and does not play active dynamical role. The Beltrami like fluctuations are short-lived and stabilize only in small and generally contiguous sub-domains that are tending to a (multi)fractal in the asymptotic limit of large Reynolds numbers, $Re \rightarrow \infty$. For the model of homogeneous isotropic turbulence the theory predicts the leading fractal dimension of **BCC** to be: $D_F = 2.5$. This

particular **BCC** is responsible for generating the Kolmogorov $-5/3$ power law energy spectrum.

The most obvious role that **BCC** play dynamically is that the nonlinear interactions in them are *relatively reduced*, due to strong spatial alignment between the velocity field $\mathbf{v}(\mathbf{r}, t)$ and the vorticity field $\omega(\mathbf{r}, t) = \text{curl}\mathbf{v}(\mathbf{r}, t)$, while the physical quantities typically best characterizing turbulence *intermittency*, such as entropy, vorticity stretching and generation, and energy dissipation are maximized in and near them. The theory quantitatively relates the reduction of nonlinear interactions to the **BCC** fractal dimension D_F and subsequent turbulence *intermittency*.

It is further asserted that **BCC** is a fundamental feature of all turbulent flows, e.g., wall bounded turbulent flows, atmospheric and oceanic flows, and their leading fractal dimension remains invariant and universal in these flows. In particular, theoretical and numerical evidence is given indicating that **BCC** in turbulent channel/pipe flows have the depth at the walls proportional to the square root of the Reynolds number in wall units, $L_y \propto \sqrt{Re}$, which is equivalent to the fractal dimension in normal to the walls y direction $D_F^y = 0, 5$, and the total dimension $D_F = D_F^{x,z} + D_F^y = 2 + 0.5 = 2.5$. Similar **BCC** structure and the same fractal dimension are suggested for geophysical turbulence, in near agreement with the recent comprehensive analysis of experimental and observational data. It is asserted that the atmospheric and oceanic events, e.g., tropical hurricanes, tornadoes and other mesoscale phenomena, and probably ocean currents are manifestations of **BCC** and their environs.

Generally **BCC** should be rather seen as the turbulence core, while the whole surrounding $3D$ flow as being created and sustained by the intense vorticity of **BCC** by means of induction, in a manner similar to that for an electric current generating magnetic field.

It is further argued that **BCC** is not only a theoretical concept important for fundamental grasp on turbulence, but may be a practical asset furnishing tools for *turbulence management in regular fluids and plasmas*.

The concept of helical fluctuations in turbulence goes 25 years back in time, and while never totally abandoned nevertheless has been residing on the fringes of research activity.

Experiment and numerical simulations had not been able to either validate or repudiate decisively the concept. However, recent large scale direct numerical simulations and proliferation of experimental and observational data showed convincingly how ubiquitous is the phenomenon of helicity fluctuations in various turbulent flows, from hurricanes and tornadoes to turbulent jets to solar wind plasma turbulence to turbulent flows in compressible fluids. This allowed a fresh look at the concept and led to a quantitative theory exposed in this paper.

The paper concludes with a brief discussion of possible similarities between turbulence and certain other complex non-equilibrium systems generating smart intrinsic coherence in the course of dissipative dynamical evolution.

Foreword

Turbulence in ordinary fluids and plasmas is one of the most typical phenomena found in observed Universe. Practically all liquids, gases and plasmas are in a state of turbulent agitation. Still turbulence remains poorly understood after hundreds of years of most intensive study by thousands of researchers. Many have resigned to the fact and abandoned the spirited debate on turbulence nature that had been typical for the second half of the 20th century. Often turbulence is perceived, by laymen and professionals alike, as chaos. In reality turbulence is organized and coherent. This truly puzzling aspect of turbulence is disregarded by many in geophysical and meteorological studies where sometimes phenomenological atmospheric models are built with only a remote reference to atmospheric turbulence organization. Far reaching predictions are made based on these models that have equally remote chances to come true.

Practitioners of turbulence in aeronautical engineering calculate the shapes and flight performances of aircrafts on a daily basis and fortunately do it successfully most of the time. As successful are mechanical and civil engineers in other disciplines dealing with flows of fluids. They recognize the coherence of turbulence since they observe routinely the so-called coherent structures in turbulent flows around airfoils and in the pipes and all other flows important for engineering applications. Nevertheless, they may be not hard pressed to come up with fundamental explanation for this coherence. The truth is that the engineering community has accumulated during the hundreds of years of experience and especially since the advent of aeronautics so much empirical data and know-how on turbulent flows in thousands of situations. All these data and tremendous know-how are summarized today in semi-phenomenological equations, tables and more recently computer models and used by engineers with remarkable dexterity and ingenuity in a reliable manner in most of engineering applications. The contemporary engineers are like the architects of the antiquity who built not burdened by the knowledge of Newton's laws and statics and did it magnificently.

The empirical knowledge of turbulence largely fails for certain applications. It has been ineffective for most attempts of the last half a century to improve turbulence management and control. No

need to have a degree in mechanical engineering, it may be even counterproductive, to notice that ocean dwellers and birds, propelling with amazing elegance through their (turbulent) habitat, have more profound understanding of turbulence management than we do. The attempts to emulate their techniques did not bring much success.

But it is when observing the atmospheric and oceanic turbulence, the planetary and mesoscale geophysical events, that the lack of understanding is most apparent. On geophysical scales we are miniscule inside observers. Even the small size geophysical events are huge for us. We don't have to peer inside trying to glimpse some small scale structures as we do in engineering scale turbulence. And here we encounter real enigmas.

The greatest one is the recurrent and obviously organized global planetary scale atmospheric and ocean flow patterns. Usually we take it for granted and do not enquire for the reasons. Many believe that meteorologists surely have the knowledge and explanations for this. In reality meteorologists know well how limited is the present understanding of the reasons for this organization. There is so much variability and chaos in every locality on Earth. Still on the global spatial and temporal scales the most basic flow patterns and subsequent weather patterns can be anticipated with great confidence. The recurrence in the global weather patterns goes on forever and we are used to it to such a degree that we don't view this truly amazing fact with surprise. We know that there are currents in the oceans and jets in the atmosphere. They flow like rivers through the ocean and air for thousands of miles for eternity of time. Why do these currents and jets not mix up, diffuse into respectively the surrounding ocean and atmosphere as we would expect based on our intuition and every day experience? Why the tropical storms, largely the same in numbers and with roughly the same intensity, are generated every year in tropical ocean regions and travel thousands of miles holding the shape and coherence to release their energy and content at about the same spots on the globe? What are tornadoes and why they strike with seasonal regularity in more or less the same regions on land?

There are many atmospheric events that we perceive with our naked eyes as organized structures in the air: cumulus clouds, squall lines, cloud streets. On a level of naive and one may say unlearn intuition we know that they are all coherent structures having shape and

made of fluid motion organized in some way, but as soon as we try to define quantitatively what is this that makes us to perceive them as such we fail. There is no adequate scientific language to serve and quantify the intuitive recognition of these coherent structures. For an honest observer not burdened with the years of study of mundane meteorological details the coherence of geophysical shapes and weather patterns, when they are considered on adequate space and temporal scales, is short of miraculous. This coherence co-exists with tremendous local variability in space and time. All meteorological modeling is ineffective when weather predictions are extended over a week time period. How is it that from all the chaos and local unpredictability of turbulent flows the tremendous global order of things in the atmosphere and oceans is created?

To understand the origins and mechanisms of global flow organization is of truly great significance. If not for the atmospheric and oceanic turbulence coherence our very existence on Earth would have not been possible. It is necessary to recognize that hurricanes, tornadoes, ocean and atmospheric currents and other magnificent geophysical events are most probably coherent manifestations of one global turbulence, rather than just local events caused by freak random coincidences of atmospheric and oceanic or land conditions.^a

Myriads of laboratory observations show the presence of what is called coherent structures (CS) in all turbulent flows, not only geophysical. No mathematical description of these structures has been developed but they appear quite obviously to experimentalist's eye as domains of seemingly structured flow patterns distinct from the surrounding flow. Every engineer and experimentalist knows that they exist because they are clearly visualized, as clouds in the sky are. Usually clouds are associated with ice particles and thermodynamic transformations resulting in rain. Few associate them in the first place with turbulence. But in fact clouds are the most obvious coherent structures in turbulence. The entire atmosphere is turbulent and the motion in the clouds is always highly turbulent, as most learn from the air travel experience. In laboratories experimentalists for better visualization inject dyes into the flow to give substance and color to coherent structures, as vapor and icicles give visual substance to clouds. Since the main concern of engineers are turbulent boundary layer (BL) flows, such as develop around airfoils or at the

walls in pipe and channel flows, the coherent structures are mostly observed and studied in laboratory conditions near the boundaries. Some experimentalists still continue linking CS to some ill understood and particular boundary effects, e.g., remains of instabilities in the incipient laminar flows, rather than intrinsic constituent elements of turbulence. With no mathematical description and no unifying physical concept the study of CS remains narrow in scope and stagnant.

Yet other kind of turbulence, which is more often than not considered quite separately from turbulence in neutral fluids, originates in plasma. In many manifestations the conducting plasma can be treated as a continuous media. In this approximation the plasma flows are described by a set of magnetohydrodynamic (MHD) equations (see Appendix). All plasma flows in astrophysical and planetary conditions are intensely turbulent, would it be the solar wind or stars corona, pulsars or galaxies. It is plasma turbulence that left unfulfilled the last 50 years of work on controlled fusion and left us without the subsequent inexhaustible source of green energy. With no fundamental understanding of turbulence and its coherent manifestations there is no much hope gaining control over it in neutral fluids or in plasma.

Despite its intractable reputation a comprehensive and to a certain extent predictable understanding of turbulence must be possible. True wealth of experimental data and numerical simulations leave no doubts that the flows of fluids, laminar and turbulent alike are adequately described by the (semi-phenomenological) Navier-Stokes equations.¹ The turbulent flows patterns as complicated as they are must be the solutions of the Navier-Stokes equations. Unfortunately the Navier-Stokes equations with the exception of few particular cases of laminar flows are analytically intractable. The attempts to apply the most sophisticated apparatus of mathematical physics developed in the last half century for other complex problems of physics, e.g., the field theories and phase transitions, all led to frustrating fiascos and only put smoke on the real issues. Much effort was spent by enthusiastic physicists and applied mathematicians trying to apply the perturbation theories, closure models and renormalization group

¹At least for the so-called Newtonian fluids such as water, gases and most of the inorganic solutions. But , for instance, polymer solutions are not Newtonian liquids.

methods to the Navier-Stokes equations with no much success. At the same time the computational capabilities have been growing in the last 25 years. Presently it is possible to simulate turbulent flows with simple geometry and at the values of Reynolds numbers approaching the laboratory conditions, although still very far from what is typical in nature. The analysis of these simulations allows making confident and deep conclusions on the structure of turbulence.

A remarkable property of the Navier-Stokes equations is that when it is re-written in dimensionless variables it contains only one intrinsic dimensionless parameter, the Reynolds number. The Reynolds number is defined as:

$$Re = \frac{\nu_L L}{\nu}, \quad (0.1)$$

where L a characteristic typical scale of the flow is, ν_L is the velocity associated with this scale and $\nu = \mu/\rho$ is kinematic viscosity of the fluid with density ρ . For almost all flows in nature the Reynolds numbers are very large, ranging from $10^4 - 10^8$ in the laboratories and engineering to $10^{10} - 10^{12}$ in geophysical phenomena and reaching fantastic values in astrophysics.² For certain critical values of $Re \geq Re_{critical}$ all laminar flows universally become unstable and seemingly erratic. This erratic or turbulent fluid motion consists of very large number of velocity harmonics or degrees of freedom.³ Therefore it is only natural that it looks chaotic. How a flow becomes turbulent from a smooth laminar one that is the problem of transition to turbulence has been to some extent understood on a basic level in the last half a century.^b

Quite distinct from the problem of transition to turbulence there is a subject of developed turbulence. Developed turbulence is a somewhat loosely defined subject. What is usually understood by developed turbulence is a complex multiscale flow pattern such as develops when the Reynolds number is much bigger than its critical value

²To have a better feeling of the orders of magnitude let us estimate a typical value of Re in the laboratory conditions for a pipe air flow: diameter of the pipe $L = 0,3 \times 10^2 cm$, $\nu_L = 10^3 cm/sec$, $\nu_{air} = 0,1 cm^2/sec$, and $Re = 0,3 \times 10^6$. For water it would be even bigger by one order of magnitude since water is much denser and $\nu_{water} \approx 0,1 \nu_{air}$.

³It will be seen shortly that the number of degrees of freedom in homogeneous isotropic model of turbulence is of order $Re^{9/4}$, a huge number for any realistic Re .

for which the transition to turbulence occurs, i.e., $Re \gg Re_{critical}$. With regard to developed turbulence it is customary to adopt the statistical language of description. The acute variability of the seemingly chaotic turbulent velocity field and very large number of participating coupled velocity harmonics, or degrees of freedom, in actual fact compels one to describe turbulence in statistical terms. Implied in the statistical description is the assumption that despite the sensitivity to initial conditions and consequent lack of dynamical predictability, the features typical for most nonlinear dynamic systems, nevertheless certain relevant quantities describing turbulent flows in a meaningful way, when obtained by averaging over a significant span of time, or over a large volume, or most generally over the ensemble of particular space-time realizations that originate from many distinct initial conditions, should acquire the same stable values (Batchelor, 1953). The same is true for the coherent structures (CS). They have best meaning within the context of developed turbulence in the sense of stable features remaining after the averaging over many realizations of turbulent flows with similar boundary conditions but originating from different initial conditions.

It should be noted that if there has been genuine progress in the last decades in understanding of fundamental developed turbulence in fluids it came from direct numerical simulations (DNS) of the Navier-Stokes equations and from fascinating geophysical observations.⁴ On the other hand physical experiment of the last decades carried out in well equipped laboratories furnished relatively negligible new insight into the fundamental structure of fluid turbulence.

In the recent large scale direct numerical simulations of turbulence in a cubic box with periodic boundary conditions (BigBox turbulence) mimicking the so-called homogeneous isotropic turbulence

⁴The DNS, i.e., numerical solutions projected on a discrete lattice pattern, but otherwise not modified Navier-Stokes equations have been pursued since the advent of supercomputers. Such simulations are difficult for two reasons: large number of independent degrees of freedom for high values of Re , of order $Re^{9/4}$, and complex boundary conditions for the flows relevant in engineering applications. Nevertheless, slowly the computational power allows reaching the values of not much smaller than in laboratory flows. DNS furnishes much more detailed information on the flow fields than experiment. DNS should not be confused with simulations of model equations. These latter are based on models of turbulence that are by default incomplete and most probably wrong.

performed by Mininni, et.al. (2008a and 2008b), it was confirmed beyond reasonable doubt that clusters of stable Beltrami like helical structures of opposite signs of helicity shaped as thin and usually prolonged vortex structures, some stretching through much of the turbulent volume, is a universal feature of developed turbulence (see **Fig. 1**). The clusters of helical structures having the opposite signs of helicity form the domains in the shape of intense vortex bands or filaments that are typical for all turbulent flows (see **Figs 2** and **3**). These domains are surrounded by the bulk of what appears as largely disorganized fluid motion.

In other recent works of Matthaeus, et.al. (2008a and 2008b), the authors experimented with numerically simulated plasma turbulence in magneto-hydrodynamic approximation (MHD). They found an anomalous distribution of angles of the plasma turbulent velocity field \mathbf{v} and plasma magnetic field \mathbf{B} convected by the velocity field. Consistently the analysis of the solar wind data from satellite measurements showed similar alignment in the live as opposed to simulated plasma turbulence. The authors point out that the alignment is similar to the alignment of velocity and vorticity that occurs in the Navier Stokes turbulence.⁵

Let us start from the Beltrami like helical structures and try to understand what is striking about their presence in turbulence. The Beltrami flows are defined as the flows where the velocity field vector is everywhere in parallel to its own vorticity, i.e., mathematically expressed as follows:

$$\mathbf{v} \cdot \boldsymbol{\omega} = \mathbf{v} \cdot [\nabla \times \mathbf{v}] = \mathbf{v} \cdot \mathbf{curl} \mathbf{v} \quad (0.2)$$

The Beltrami flows are well known mathematical curiosities. They are exact solutions of the inviscid Euler equations describing inviscid (and in that unrealistic) flows, but having certain fascinating mathematical properties that were exposed in the works of Arnold (1965,

⁵Solar wind is well conducting turbulent plasma blown from Sun. The magnetic field in turbulent plasma is convected by the turbulent velocity field in a manner not dissimilar to convection of vorticity field generated by this velocity field. Although the analogy is not at all total, since \mathbf{B} is an extraneous to \mathbf{v} field while $\boldsymbol{\omega} = \mathbf{curl} \mathbf{v}$, it is nevertheless profound in certain manifestations. In particular, the observed alignment between \mathbf{v} and \mathbf{B} in turbulent plasma is most probably similar in nature to the alignment between $\boldsymbol{\omega}$ and \mathbf{v} in turbulent flows of neutral fluids.

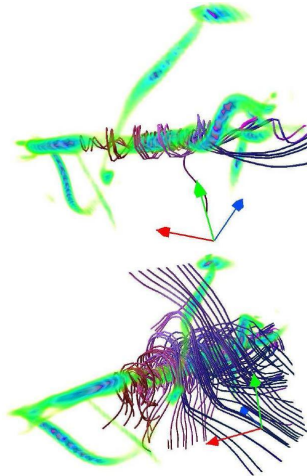


Figure 1: It is stated in Mininni, et. al. (2008a) that this is a zoom on a sample region of high vorticity in the flow. Turbulence in realizations is very inhomogeneous or intermittent in its structure and the regions of high vorticity occupy relatively small part of the total flow domain. Typical vortical structures, actually two merging vortical structures in the South-West corner, are sampled from this region with the velocity field lines drawn inside the structure in the upper image and near the structure in the image below. The authors claim that the velocity field lines are strongly helical aligning with the vorticity field lines.

1966 and 1974), but clearly understood by Moffatt (1985). In particular, Beltrami flows have the maximal normalized helicity value for given \mathbf{v} and $\boldsymbol{\omega}$.^c Helicity is a topological invariant, a particular consequence of the Kelvin's conservation of circulation theorem in inviscid flows. In a simplified interpretation of all vortex lines closed helicity is a well known topological Hopf invariant, an integer measure of knottedness of divergence free solenoidal vector field lines and in this case vorticity field lines.⁶ But generally helicity is a continuous

⁶More precisely Hopf topological invariant is easy to visualize. It arises for-

invariant and more complex topological measure defined in a rigorous mathematical context in Arnold (1965 and 1966).

In the flows of real viscous fluids that are subject to the Navier-Stokes equations, for which neither energy nor helicity are invariant anymore because of the molecular scale viscous dissipation, still there are flows analogous to Beltrami flows which are the exact dynamical solutions. These are exponentially decaying with time Beltrami flows that relax asymptotically with time to the state of equilibrium still fluid at each point and therefore retaining in some sense the invariant topology of vortex lines, e.g., the normalized helicity (Libin, et.al., 1987; Libin, 2008).

How would such very particular and obviously coherent flow patterns in a well defined mathematical sense, preserving their distinct shape and topology appear in the midst of highly variable in space and time and seemingly disorganized fluid motion? Homogeneous isotropic turbulence is a pure turbulence model not contaminated by extraneous complications such as solid boundaries or complicated thermal sources typical for geophysical turbulence. As such it comes explicitly under the purveyance of the concepts first clearly expressed by Richardson in 1922 and culminating in the Kolmogorov theory of turbulence formulated in 1941, which since then has dominated the fundamental understanding of turbulence

The Kolmogorov theory postulates that the evolution of turbulence from any and all initial flow conditions is a statistical hierarchy of homogeneous isotropic and structureless eddies with progressively decreasing spatial scales randomly filling the whole fluid volume. The main feature of these eddies is a steady state constant flow of energy cascading from the larger scale eddies to the smaller ones with the smallest scale eddies benignly dissipating energy through molecular viscosity into heat. The main prediction of Kolmogorov theory, the $-5/3$ power law energy spectrum for the turbulent velocity fluctuations as a function of inverse scale (or wavenumbers in Fourier space),

mally from the mapping of 3 - dimensional sphere into a 2 - dimensional sphere, $S^3 \rightarrow S^2$. There is a family of invariants in higher dimensions $S^{n+1} \rightarrow S^n$ and they are quite often considered in various problems in physics, although they are not associated with knots that are specific to $3D$ space. For instance, the closed line thread like vortices in superfluid helium, which essentially as nearly an ideal fluid as can be in nature, are characterized by a simple integer topological invariant-the winding numbers-generated by the mapping $S^2 \rightarrow S^1$.

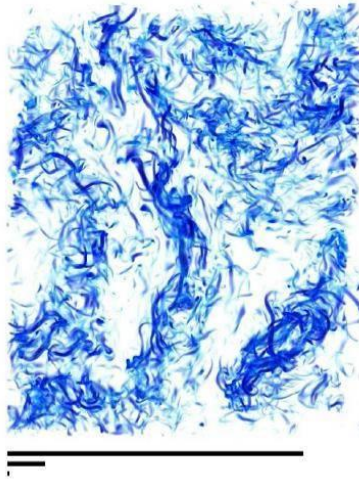


Figure 2: Borrowed from Mininni, et. al. (2008b) it shows an extended image of a flow sub-domain in DNS of turbulence with periodic boundary conditions. The authors choose a sub-domain in such a way that everywhere in it vorticity is large by comparison with its own average, $|\omega| > 6 < |\omega| >$. The shown bars indicate the dimensions of three typical scales: the largest integral scale, nearly the periodic Box size, the so-called Taylor microscale (defined in such a way that it is always supposed to be in the inertial range) and the smallest scale at which viscous dissipation is definitely dominant. One can see that the high vorticity is organized into extended vortical bands that themselves tend to bunch together into clusters. Although the bands and the clusters of high vorticity reach the largest scales in one dimension their total volume is a small fraction of the total turbulent flow volume. Altogether the sub-domains of high vorticity are a small fraction of the total flow domain.

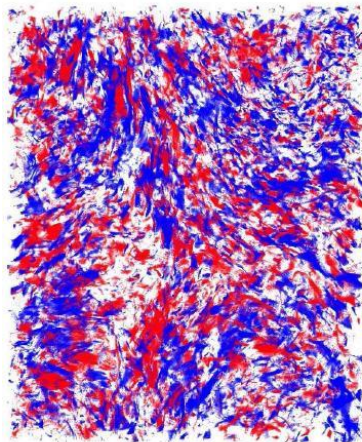


Figure 3: From the same work of Mininni, et. al. (2008b) it shows the same flow sub-domain as in Fig. 2, but in violet and red colors is depicted the helical vortices with opposite sign of helicity. Each of these vortices is similar to the one from Fig. 1. As the authors claim only the regions of near to total helical alignment between velocity and vorticity are painted. Clearly the vortical structures in Fig. 3 almost totally overlap the bands of high vorticity in Fig. 2. Thus the vorticity bands in Fig. 2 are largely made of the helical cells. Important is to note that the total helicity of the sub-domain is near to zero. This shows the power of helicity fluctuations concept as opposed to average helicity. In each of these cells the alignment is strong and helicity is near to maximal possible value for the given absolute values of velocity and vorticity in each of the cells. In other words they are nearly Beltrami cells.

has been approximately observed in great many experiments and geophysical observations and therefore seems reasonably confirmed, even though there is still lack of confidence as regards the experimental accuracy.^d But the large scale strongly anisotropic coherent helical patterns have no place in Kolmogorov turbulence; they are in total contradiction with the usual wisdom that postulates that turbulence is made of volume filling structureless eddies.

To be more precise physical experiment, DNS in the last decades, and of course the atmospheric observations have been long time revealing that the distributions of small distance variations of the flow field velocity, vorticity, viscous energy dissipation and other related quantities⁷ are statistically inhomogeneous and strongly non-Gaussian, in contrast with the velocity field at a point which is primarily determined by large scale motion and well approximated by the Gaussian law statistical distribution. It has been long known, e.g., Batchelor and Townsend (1949) and Batchelor (1953) that the anomalously large, by comparison with the Gaussian statistical law values, vortical activity and energy dissipation are situated in bands, filaments as they are often called, in relatively small flow sub-domains and temporal bursts of activity. This bunching together in small space sub-domains and short temporal bursts of intense activity of small scale dominated turbulent events is known as turbulence intermittency (see Section 5 below). This phenomenon gave rise to the phenomenological fractal models of turbulence, e.g., Kolmogorov (1962), Obukhov (1962), Novikov and Stewart (1964), Monin and Yaglom (1975), Mandelbrot (1974), Mandelbrot (1983), etc. These fractal models of turbulence as a rule saw intermittency as corrections to the Kolmogorov turbulence affecting high order statistical correlations but not undermining the basic tenets of the cascade theory.⁸

Intermittency is not a phenomenon found in turbulence alone. It

⁷That is essentially the quantities expressed via velocity field space and time derivative

⁸In particular, fundamentally important for understanding the fractal structure of atmospheric turbulence are the works of Shaun Lovejoy and Daniel Schertzer and co-workers spanning the period of 25 years of atmospheric observations and theoretical deductions. Their works will be cited when appropriate below, but I would like to mention that now only few would doubt the multifractal vision of all important atmospheric and more generally geophysical structure and events and their importance for forecast and understanding of the atmosphere and ocean.

is a feature typical of dynamical chaos originating in most nonlinear dissipative systems. But such endemic intermittency is of course a far cry from the spatially delineated helical coherent structures that are observed in the midst of turbulence so clearly and unambiguously. The coherent helical build-up of the intermittent bands of vortical activity in the midst of seemingly disorganized fluid motion, in a model that is closest in mimicking the conditions of homogeneous isotropic turbulence, shows that it is timely, in fact long overdue, to reassess both the Kolmogorov theory of turbulence and the still prevailing feeble interpretation of coherent structures in turbulence.

The observations of Beltrami like helical structures and related anomalies, made in Mininni, et.al.(2008a) and Mininni, et.al.(2008b), were largely predated and predicted in the works of previous decades. The theoretical concept of helical structures or fluctuations and their connection to the coherent structures (CS) on one hand and intermittency and fractals on the other, in turbulent flows irrespectively of their origin, was formulated in a series of works published over a period of time, e.g., Levich (1982), Levich and Tsinober (1983), Tsinober and Levich (1983), Levich and Tsinober (1984), Levich and Tzvetkov (1984 and 1985), Shtilman, et.al. (1985), Levich (1987), Levich and Shtilman (1988), Levich, et.al. (1991), Levich (1996). In these works the viewpoint was advanced that the coherent helical fluctuations are the fundamental building blocks of all turbulent flows. The fluctuations are borne and die at multiple scales, as befits the general self-similarity and scaling conceptual foundation of turbulence theory, and on a fast time scale that tends to zero in the limit $Re \rightarrow \infty$ in most of turbulent flow space. Because of this fast time scale the helicity fluctuations were called 'virtual'. The helical fluctuations are of both positive and negative helicity signs, clustering and screening each other, so that the global helicity in representative space/time domains remains small, but strictly speaking with mirror symmetry likely not respected.⁹

⁹Althoughh certain violation of global mirror symmetry was observed in DNS and experiment, e. g., see the references in a footnote below. Nevertheless, the phenomenon is entirely different from the one caused by a non-zero global helicity in the flow. Such global helicity is of little relevance for the fine structure of turbulence and can be disregarded in most cases. The role of spontaneously generated global helicity is not discussed here although the effect is most probably there.

In each of the virtual helical fluctuations the nonlinear coupling between the turbulent harmonics is somewhat reduced during their life time, while the vorticity and vorticity production attain increasing values at their boundaries. The volume occupied by the helical fluctuations decreases as most of them die out because of the short life span of helical correlations but eventually the fluctuations attain stability, or better to say a long enough life time span on a fractal sub-domain with the material volume tending to zero in the limit $Re \rightarrow \infty$. But at the same time despite their small relative volume as compared with the total flow domain the helicity fluctuations and their environs contain most of turbulent activity.

The assumption of self similarity implies that while the volume occupied by the helical structures tends to zero in the limit $Re \rightarrow \infty$, their total bounding surface area tends to infinity. The regions of decreased nonlinear coupling and on the contrary most intensive turbulent fields amplitudes, e.g., vorticity generation, overlap each other in this limit. In other words the helical structures and their boundaries together become a fractal set of dimension $2 < D_F < 3$ embedded in the 3D fluid domain.

If the transient fluctuations and stable helical structures in thus defined sub-domains of flow realizations are indeed present than the first and simplest experiment would be to measure the distribution of angles between \mathbf{v} and $\boldsymbol{\omega}$ fields. The distribution is likely to show tendency for \mathbf{v} and $\boldsymbol{\omega}$ to be parallel or anti-parallel. Indeed, in the pioneering DNS of Shtilman, et.al. (1985), Pelz, et.al. (1985) and Pelz, et.al. (1986) such anomalous alignment of \mathbf{v} and $\boldsymbol{\omega}$ was observed as moderate size peaks in the probability distribution function $Pdf(\cos\theta) = Pdf(\mathbf{v} \cdot \boldsymbol{\omega} / |\mathbf{v}||\boldsymbol{\omega}|)$ near the maximal and minimal values $\cos\theta = +1$ and $\cos\theta = -1$. The peaks are natural to interpret as a contribution from the helical fractal set to the total $Pdf(\mathbf{v} \cdot \boldsymbol{\omega} / |\mathbf{v}||\boldsymbol{\omega}|)$ in space/time realizations, but also as a relict trace from the transient helical fluctuations (see Fig. 4). This is why this "anomaly" persists typically in representative sub-domains of turbulent flow, including the domains with low turbulent intensities.¹⁰

¹⁰The anomalous angle distribution was first numerically established at the very dawn of the supercomputers era and emerging numerical studies of the Navier - Stokes equation, in Shtilman, et al. (1984) for decaying Taylor - Green vortex and in Pelz, et al. (1985) for the channel flow turbulence. Many at the time saw it as an artifact of numerical simulations. Since then the alignment was

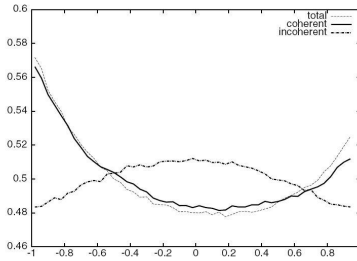


Figure 4: From Farge, et. al. (2001) it shows the $Pdf(\mathbf{v} \cdot \boldsymbol{\omega} / |\mathbf{v}||\boldsymbol{\omega}|) = Pdf(\cos\theta)$ in DNS of turbulence in a box with periodic boundary conditions. In this interesting DNS the authors using a mathematical method of wavelets sampled out what they call a coherent component of the velocity field in turbulent flow. This coherent component shows a quite distinct alignment of \mathbf{v} and $\boldsymbol{\omega} = \text{curl}\mathbf{v}$ fields. The incoherent component shows no alignment. But apparently the total velocity field shows similar alignment. What it means likely is that the regions of low vorticity typically retain certain coherence as well and can be seen as a relict trace of previously strongly coherent helicity fluctuations. The Pdf results are totally similar to the ones obtained in much earlier works of Shtilman, et. al. (1985) and Pelz, et. al. (1985).

It is likely that for similar reasons and to the same end there is an observed anomalous alignment between the velocity field and magnetic field in turbulent plasma described in Matthaeus, et. al. (2008a and 2008b) for DNS of MHD turbulence and actual observations of real turbulence in solar wind (see Fig. 5 and Fig. 6). The authors reasonably asserted that the cross-helical patches similar to fluid turbulence helical clusters exist in plasma turbulence as well, but with

confirmed in many DNS, e.g., in Pelz, et al. (1986), Levich and Shtilman (1988), Farge, et. al. (2001). The alignment to some extent was also confirmed in unique experimental studies first initiated by A. Tsinober and E. Levich. The excruciatingly difficult measurements of vorticity and alignment were reported in Kit, et.al. (1987, 1988a and 1988b), but caused soem controversy, Wallace and Balint (1990). The experiment were followed by Kholmiansky, et. al. (1991) and Kholmiansky, et al. (2001).

substitution of vorticity by the magnetic field.

The reduction of non-linear coupling in the hierarchy of helical fluctuations is of great significance and allows in the long run turbulence in fluids as we see it in nature to exist (Levich, 1987; Shtilman and Polifke, 1989). It is the mechanism of intermittency and the reason for the relative stability of large scale vortical structures in the flows of fluids in nature. It was asserted in Levich (1987) that a spatially and temporally small active sub-domain formed by the hierarchy of helical fluctuations generates the Kolmogorov spectrum. This assertion allowed calculating the reduction of the nonlinear coupling in the structures in this sub-domain and subsequently the fractal dimension of this sub-domain. The calculation led to $D_F = 2.5$ in the limit $Re \rightarrow \infty$. Although the Kolmogorov energy spectrum is correct the mechanism that forms it is different from the one postulated in the Kolmogorov theory. Subsequent to the helical build-up of turbulence it was proposed in Levich (1980) and Levich (1987) that the interactions in turbulent flows are inherently non-local with the flow harmonics of widely disparate scales strongly interacting with each other. This is again contrary to the tenets of Kolmogorov theory, but strongly supported by the cited DNS.^e

Somewhat similarly with the above concept Moffatt (1985) and Moffatt (1986) suggested that turbulence generally consists of unstable/shortlived in space/time Beltrami blobs separated from each other by the surfaces of intensive dissipation. Moffatt (1984) also suggested that the Kolmogorov energy spectrum is formed not by the Kolmogorov mechanism, but via induction by a particular class of vortical spiral singularities embedded in turbulence midst. His analysis and classification of the unique properties of the Euler flows (stationary solutions of the ideal fluid Euler equations) and in particular of Beltrami flows is imperative for understanding the helical concept of turbulence.

Over two and a half decades have passed since the first publications on the helical concept of turbulence. And only now the experimental mass of evidence and DNS brought back a positive momentum to studying this concept. While the proof of presence of coherent helical structures seems uncontested after the works cited above their origin and the fundamental role they play in turbulence remain to be further explained.^f

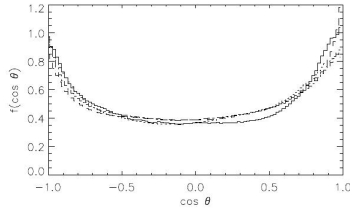


Figure 5: From Matthaeus, et. al. (2008a) it shows $Pdf(\mathbf{v} \cdot \mathbf{B}/|\mathbf{v}||\mathbf{B}|) = Pdf(\cos\theta)$ for DNS of MHD turbulence in a box with periodic boundary conditions. The different curves correspond to different runs with different initial and forcing conditions. The qualitative similarity with Fig. 4 is pretty obvious.

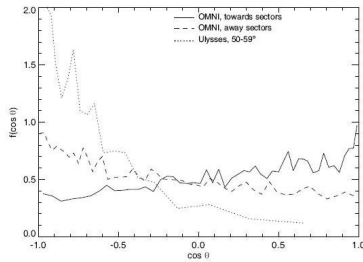


Figure 6: The same as in the previous figure, but for satellite measurements over long time periods and different orientations in relation to the solar wind. It is clear that after averaging over all orientations the resulting figure would look strikingly similar to the one obtained from DNS (for instance combine away and towards sectors).

The assertion reiterated herein is that turbulent flows in laboratory and in nature, in boundary layer flows, in atmosphere and oceans, in neutral fluids, and probably in electrically conducting plasmas, are made of helicity (or in the case of MHD plasmas cross helicity) fluctuations in various stages of birth, evolution and disintegration. The all scale helical structures of opposite sign are not a cor-

recting feature superimposed on Richardson-Kolmogorov turbulence structure, imagined as made of randomly dispersed and structureless fluid eddies, *they are rather the turbulence*. And long life time stability of relatively large scale helical blobs bunched into small sub-domain is made possible by diminishing and balancing the nonlinear coupling in the Navier-Stokes equations that creates them and strives to destroy. Although the long-lived helical structures occupy only a vanishingly small fraction of the fluid space/time domain they cluster together like cells with opposite sign of helicity forming large patches of the most intensive events, e.g., coherent patches of vorticity and vorticity generation that are usually observed as CS. It is this domain that is asserted to sustain the Kolmogorov energy spectrum and turbulence in general in the whole fluid domain.

It is almost trivial to add that the solid boundaries most actively facilitate the coherent helical structures birth and death, e.g., Levich (1996), since in general turbulence originates most readily and is most intensive in the boundary layer (BL) near the solid boundaries. The CS in BL turbulence are just the elongated domains of intense vorticity and it was conjectured that they are made of strongly helical cells of opposite sign of helicity, so that the total helicity is zero or small, but with reduced nonlinear coupling inside each cell providing relative stability to them and their environs. Equally they exist as elongated vortex bands in turbulence far away from boundaries and in turbulence created with maximal nearness to (statistically) homogeneous and isotropic conditions, e.g., turbulence past a grid and in BigBox numerical turbulence as is clear from the cited works, and may be most importantly in atmospheric, oceanic and plasma turbulence. To avoid semantic ambiguity and intermittent choice of words patches, bands, or filaments, I suggest to call the coherent structures by what they really are - Beltrami cellular clusters - **BCC**.

The above assertions were in part made before. In what follows they will be reiterated and further quantitatively justified with confidence supported by accumulated experimental evidence.⁹

1 Basic Equations and Definitions

This work is directed not necessarily to experts in fluid mechanics but to a more general academic audience. Turbulence is so an

amazing phenomenon that science oriented audience of general profile deserves to be updated on progress in its understanding. This is why it is prudent to start from the beginning.

The basic equations governing the flows of incompressible viscous fluids are the Navier-Stokes equations:

$$D\mathbf{v}/D\mathbf{t} = \partial_t\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} == -\nabla P + \nu\Delta\mathbf{v} + \mathbf{F}, \quad (1.1)$$

or in index notations useful for anisotropic flows:

$$\partial_t v_i = \partial_k(v_i v_k + \delta_{ik}P) = \nu\partial_k\partial_k v_i + F_i \quad (1.2)$$

In the Eqs. (1.1) - (1.2) P is pressure, F_i is an external body force acting on the fluid assumed to be a divergence free ($\nabla \cdot \mathbf{F} = \partial_i F_i = 0$), $\partial_t = \partial/\partial t$; $\partial_i = \partial/\partial x_i$; $\nabla^2 = \partial_i^2 = \Delta$ is the Laplacian operator, ν is a kinematic fluid viscosity, the density ρ is set to unity and the fluid is assumed incompressible, so that the velocity field is divergence free:

$$\nabla \cdot \mathbf{v} = \partial_i v_i = 0. \quad (1.3)$$

The quadratic nonlinear term in The Navier-Stokes equations consist of two parts. One is the inertial convection term from the general definition of the full derivative $D/Dt = \partial_t + \mathbf{v} \cdot \nabla$, in the Eulerian (field) representation of continuous media.¹¹ The other is the obvious pressure gradient force, which is in incompressible fluids not an independent field, but is determined from Eq. (1.3). This is clear by applying the divergence ∇ operator to the both sides of Eq. (1.1) with the result:

$$P = -\Delta^{-1}\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}]. \quad (1.4)$$

Together with boundary conditions, for instance the usual for applications no-slip boundary conditions at the solid surfaces bounding the flow:

$$\mathbf{v}_S(\mathbf{r}, \mathbf{t}) = \mathbf{0}. \quad (1.5)$$

The Eqs. (1.1) (1.4) in principle fully define the velocity field and pressure would it be laminar or turbulent flow.^h If to set $\nu = 0$, a

¹¹In the Eulerian representation a fluid flows in relation to an observer so that

$$D/Dt = \partial_t + v_i\partial_i = \partial_t + \mathbf{v} \cdot \nabla$$

physically impossible situation that would correspond to the infinite molecular path length, than the Navier-Stokes equation becomes the Euler equations. The latter are the Newton equations for the continuous media driven by the pressure gradient moving in relation to an observer:

$$D\mathbf{v}/D\mathbf{t} = \partial_t \mathbf{v} + \mathbf{v}(\nabla \cdot \mathbf{v}) = -\nabla P. \quad (1.6)$$

Also in this approximation, which is never true in reality, the non-slip boundary condition does not hold since there is no friction at the boundaries. Instead a less restrictive boundary condition is appropriate stating that the normal to the surface boundaries velocity component is zero should be used, i.e., $\mathbf{n} \cdot \mathbf{v}_S(\mathbf{r}, \mathbf{t}) = \mathbf{0}$.¹

It is instructive to rewrite the Navier-Stokes equations in the so-called rotational form. Setting the non-fundamental force $\mathbf{F} = \mathbf{0}$ for the moment and using identical vector transformations the equivalent form for the Eq. (1.1) is as follows:

$$\partial_t \mathbf{v} - [\mathbf{v} \times \boldsymbol{\omega}] = -\nabla(P + \mathbf{v}^2/2) + \nu \Delta \mathbf{v}, \quad (1.7)$$

or in tensorial form:

$$\partial_t v_i - \epsilon_{ikl} v_k \omega_l = -\partial_k (P + v_i v_k / 2) \delta_{ik} + \nu \Delta v_i, \quad (1.8)$$

because of the identity:

$$\partial_k (v_i v_k) \equiv -\epsilon_{jkl} v_k \omega_l + \delta_{ik} \partial_k (v_i v_l) / 2, \quad (1.9)$$

where $\boldsymbol{\omega} = [\nabla \times \mathbf{v}] = \mathbf{curl} \mathbf{v}$, or in the components $\omega_i = \epsilon_{ikl} \partial_k v_l$ is the vorticity field introduced in Foreword. Applying the *curl* = $\nabla \times$ operator to both sides of Eq.(1.4) we obtain even more compact looking form of the Navier-Stokes equations:

$$\partial_t \boldsymbol{\omega} - \mathbf{curl}[\mathbf{v} \times \boldsymbol{\omega}] = \nu \Delta \boldsymbol{\omega}, \quad (1.10)$$

or in another useful form the Eq.(1.10) can be written as follows:

$$D_t \boldsymbol{\omega} = \partial_t \boldsymbol{\omega} + (\mathbf{v} \cdot \nabla) \boldsymbol{\omega} = -(\boldsymbol{\omega} \cdot \nabla) \mathbf{v} + \nu \Delta \boldsymbol{\omega}. \quad (1.11)$$

In principle for a given vorticity field the velocity field $\mathbf{v} = (\nabla \times)^{-1} \boldsymbol{\omega}$ is fully defined everywhere in fluid domain, calculated by means of an

integral expression similar to Bio-Savart relation between the magnetic field and the current creating it, or the other way around.¹² Therefore the Eq. (1.10) is a closed equation for the vorticity field.

This way of rewriting the Navier-Stokes equation is important and revealing for understanding fluid dynamics. In the approximation of ideal fluids $\nu = 0$ the equation for ω becomes one from a wide class of dynamical equations for the so-called "frozen-in" fields convected by a fluid motion. Frozen-in fields can be scalars, vectors, tensors or anything else that defines a field. In fluid mechanics the frozen-in fields are usually scalar and vector fields. If a frozen vector field line at zero time passes through a certain material fluid line belonging to a fluid element then for all other times it will pass through the same material line. The vector lines are rigidly attached to the fluid particles through which they pass. If the fluid elements are running away from each other, as they do in a turbulent (or any chaotic) flow then the corresponding frozen field lines will be stretched indefinitely to follow the motion of fluid particles. The nonlinear term in the left side of Eq. (1.11) is actually a passive convection of the field lines, while the nonlinear term on the right side of Eq. (1.11) is responsible for the stretching of the vector field line. If one considers a scalar field then the analogous equation of motion will have only a convection term left. For instance the dynamical equation for the heat transfer is:

$$\partial_t T + \mathbf{v} \cdot \nabla) \mathbf{T} = \kappa \Delta \mathbf{T}, \quad (1.12)$$

where κ is the molecular heat conduction coefficient. When $\kappa = 0$ Eq. (1.12) describes a passive scalar convection. Even such simplest situation becomes very complicated when the velocity field is turbulent or merely a given random field (e.g., Falkovich and Fouxon, 2005).¹³

¹²One should not forget the potential part of the velocity field that should be restored to have the incompressibility condition $\nabla \cdot \mathbf{v} = \partial_i \mathbf{v}_i = \mathbf{0}$ fulfilled (e.g., Batchelor, 1979). In practical applications the solution of Eq. (1.6) and restoration of the potential part of the velocity field from the corresponding Laplace equation is difficult for most boundary conditions, except infinite media and periodic boundary conditions. Nevertheless, the fact that the vorticity ω field can be seen as creating the flow \mathbf{v} field is very significant for understanding turbulence.

¹³Convection of passive admixtures is a complex problem it nevertheless seems tractable. In part this may be the reason why it has become very popular in recent years. However it unfortunately does not take us much closer to understanding turbulence, or even the probably simpler problem of convection of vector fields,

An important and much more complex example is the magnetic field \mathbf{B} convected by a flow of conducting plasma. If the plasma velocity is stationary than the whole set of MHD equations degenerates into a dynamical equation for the magnetic field and is as follows (see Appendix):

$$\partial_t \mathbf{B} - \mathbf{curl}[\mathbf{v} \times \mathbf{B}] = \eta \Delta \mathbf{B}, \quad (1.13)$$

where η is the electric conduction coefficient. When η is substituted by ν and \mathbf{B} by $\boldsymbol{\omega}$ the Eqs. Eqs. (1.13) and (1.10) are formally the same. When $\nu = \eta = 0$, respectively in (1.10) and (1.13), the fields are frozen in the fluid and their lines move together with the material fluid elements to which they are attached at $t = 0$. There is a profound analogy between the kinematic properties of frozen-in vector fields. The study of frozen-in fields goes all the way back to the 19th century Kelvin's theorem of conservation of circulation or what is the same the theorem of conservation of vorticity flux and similar theorem of conservation of magnetic flux. More recently there has been renewed interest to the frozen-in fields since their properties are closely connected to mathematical knots and other topologically invariant objects. With respect to fluid mechanics this subject was analyzed by Arnold (1965, 1966), but deeply understood and made clear by Moffatt (1985).

The dynamics of frozen-in fields is peculiar even for the simplest case of a scalar additive convected by a random velocity field. But for the vector fields it is extremely rich in results. For instance the dynamo effect, the exponential growth of magnetic field in moving conducting fluids, so important for the origin of magnetic field in astrophysics and elsewhere, is a result of this peculiar frozen-in dynamics.¹⁴ In a random flow field the fluid particles generally "run away" from each other like in Brownian motion, so that the distance between any pair of them increases with time. In consequence the magnetic field lines stretch by these "run away" fluid particles because the lines are frozen into the fluid particles. The stretching of

e. g., the magnetic field, by turbulent velocity field (e. g., Moffat, 2001)

¹⁴In this case the flow field should not be necessarily the real turbulent flow field as it develops from the Navier-Stokes equation but some stationary random field with non-zero helicity coarsely modeling the integral properties of the turbulent flow field. Eventually only the integral helicity turns out to be a relevant parameter for the dynamo growth of the magnetic field.

magnetic field lines is necessary (but not sufficient) condition for the growth of magnetic field.^j The terms in (1.11) and (1.13) responsible for the field lines stretching are $-(\boldsymbol{\omega} \cdot \nabla)\mathbf{v}$ and $-(\mathbf{H} \cdot \nabla)\mathbf{v}$, respectively for $\boldsymbol{\omega}$ and \mathbf{H} field lines.

The difference between Eq. (1.13) and Eq. (1.10) is that the magnetic field is convected by a given fluid motion, so that Eq. (1.13) is quasi-linear, but the vorticity field is the *curl* of the velocity field and hence Eq. (1.10) is fundamentally nonlinear. In fact for a given distribution of vorticity the velocity field can be determined everywhere by means of induction from an integral relation similar to Bio-Savart induction law relating current and the magnetic field it generates (e.g., Batchelor, 2000). In other words the vorticity field can be seen as a principal one generating the flow field such that the vortex lines are frozen in this flow in ideal fluids. In real turbulent motion the fluid particles also "run away" from each other, so that the distance between each two initially closely adjacent fluid particles grows exponentially with time for short times before the nonlinearity effects take hold. The vortex lines are stretched by this run away motion since they are frozen into the fluid particles. This stretching of vortex lines results in the rapid growth of vorticity amplitude and is generally believed to be the basic mechanism of developed turbulence formation in fluids. Despite the differences the kinematic similarities between the MHD and the Navier-Stokes equations have profound methodological implications for the theory of turbulence and therefore I will discuss this analogy and the implications below.

Since Re defined by (0.1) is the only intrinsic dimensionless parameter in the Navier-Stokes equations the flows of identical geometry but with different velocities, or integral scales, or viscosity are fundamentally the same as long as Re remains the same. This is the famous self-similarity true for all flows. This is why when we consider a steady pipe flow for instance we can be confident that it is enough to consider just one steady pipe flow and then it will be the same for all the other steady pipe flows with the same Reynolds number value.

As was pointed out in Foreword for most of the flows in nature the Reynolds numbers are very large, with notable exception of capillary and some biological flows. Thus practically all flows are very strongly turbulent. It seems tempting then to seek a theory of turbulence in the limit of $Re \rightarrow \infty$. The self-similarity than indicates that

in the limit $Re \rightarrow \infty$, the flow properties should become Re independent, i.e., universal as long as the boundaries and geometry remain the same or their influence can be disregarded. Of course this does not mean that one can just omit the viscous term in Navier-Stokes equations and use the ideal fluid description. The limit $Re \rightarrow \infty$ is not at all the same as $\nu = 0$. Because the viscous term is the higher derivative term and there are always such small scale spatial variations of the turbulent velocity field when it becomes dominant. This may be a trivial comment that nevertheless is important to bear in mind. From the outset it should be noted that it is not at all granted that the universal limit $Re \rightarrow \infty$ exists. Or it may exist for some velocity related quantities and not to exist for others. In any event it is instructive to rewrite the Navier-Stokes equations in dimensionless units as follows:

$$X_i = x_i/L; V_i = v_i/v_L; P' = P/v_L^2; \tau = \nu t/L^2. \quad (1.14)$$

Then the Eqs. (1.1) are as follows:

$$\partial_\tau V_i + Re \partial_k (V_i V_k + \delta_{ik} P') = \partial_k^2 V_i. \quad (1.15)$$

The Eqs. (1.1) can be also rewritten in a different way such that the Re^{-1} factor stands in front of the viscous second space derivative term in the r.h.s. of the Navier-Stokes equations. But the way it is written in (1.15) is more suitable for the further exposition.¹⁵

There are (at least) three quantities that are especially important for the theory of turbulence: energy (per unit mass), helicity and vorticity. Let us start from the energy per unit mass (it is reminded that the fluid density is set to unity, $\rho = 1$). Multiplying Eq.(1.2) by v_i and integrating over the fluid volume one finds after some simple rearrangements ($i = 1, 2, 3$ and $\partial_k = \partial/\partial x_k$):

$$\begin{aligned} \partial_t E = \partial_t \int v_j^2/2 dV = \oint [v_k(v_i^2/2 + P) - 2\nu v_i e_{ik}] dS_k + \\ -2\nu \int e_{ik}^2 dV + \int v_i F_i dV, \end{aligned} \quad (1.16)$$

¹⁵In what follows we shall alternate dimensionless and dimensional units without changing the notations when convenient.

where the integrand of the surface integral is the energy flux:

$$j_k^E = v_k(1/2v_i^2 + P) - 2\nu v_i e_{ik}, \quad (1.17)$$

where e_{ik} is the stress (or deformation) tensor typical for continuous media:

$$e_{ik} = 1/2(\partial_k v_i + \partial_i v_k). \quad (1.18)$$

The remaining two terms on the r.h.s. of Eq. (1.16) are the energy increasing due to the work done by the body force \mathbf{F} and the energy decreasing due to dissipation by viscous forces. If the surface is a solid boundary, at which $\mathbf{v} = \mathbf{0}$ the surface integral vanishes, the nonlinear coupling term naturally conserves energy, and one is left with the energy acquisition due to the external force balanced by the dissipation terms. If one sets both dissipation and the force zero than the Eq. (1.16) is the obvious energy conservation law in inviscid fluids. The local viscous dissipation rate per unit volume per unit time is hence:

$$\epsilon(\mathbf{r}, \mathbf{t}) = -2\nu(\mathbf{e}_{ik})^2 = -\nu/2(\partial_k \mathbf{v}_i + \partial_i \mathbf{v}_k)^2. \quad (1.19)$$

Apart from energy a quantity fundamental for turbulence description is a quadratic scalar measure of vorticity intensity-entropy $\Lambda = \omega^2$. One can derive from Eq.(1.11) the following balance equation:

$$\int \partial_t \Lambda / 2 dV = \int \omega \cdot (\omega \cdot \nabla) \mathbf{v} - \nu \int [\nabla \times \omega] dV. \quad (1.20)$$

In (1.20) the integration is assumed over the infinite volume or a compact flow domain. In a steady state it follows then:

$$\int \omega \cdot (\omega \cdot \nabla) \mathbf{v} dV = \int \omega_i \omega_k \nabla_k v_i dV > 0. \quad (1.21)$$

The meaning of (1.21) is that the nonlinear coupling while conserving energy generally increases the global entropy at the same time through the vortex lines stretching. It will not only for certain configurations of vorticity field and in particular for such that are solutions of the inviscid Euler equations and thus cannot be in a stationary equilibrium with viscous dissipation. The relation (1.20) is one of a few exact relations that can be derived from the Navier-Stokes equations and therefore is of special value. It is easy to see also with some

trivial vector manipulations that the viscous dissipation term in the balance equation (1.16) can be rewritten in the following identical way:

$$-2\nu \int e_{ik}^2 dV = - \int \omega^2 d\mathbf{V}. \quad (1.22)$$

This is an important relation showing that as enstrophy grows simultaneously the global energy viscous dissipation does (note that of course $2e_{ik}^2 \neq \omega^2$ locally).

Helicity is another peculiar invariant of the inviscid Euler equations. Let us introduce helicity density pseudo-scalar product $h = \mathbf{v} \cdot \boldsymbol{\omega}$ and helicity H as follows:

$$H = \int h dV = \int \mathbf{v} \cdot \boldsymbol{\omega} d\mathbf{V}. \quad (1.23)$$

Then a simple calculation shows that:

$$\begin{aligned} \partial_t H &= \partial_t \int h dV = \partial_t \int \mathbf{v} \cdot \boldsymbol{\omega} d\mathbf{V} = \\ &= \oint [\omega_k (1/2v^2 - P) - v_k h] dS_k - 2\nu \int \omega_i \epsilon_{ijl} \partial_j \omega_l dV, \end{aligned} \quad (1.24)$$

($\omega_i \epsilon_{ijl} \partial_j \omega_l = \boldsymbol{\omega} \cdot \mathbf{curl} \boldsymbol{\omega}$, ϵ_{ijl} is the absolutely anti-symmetric unit tensor). The integrand in Eq.(1.24) is the helicity flux similar to the energy flux (1.17):

$$j_k^H = \omega_k (1/2v_i^2 - P) - v_k h. \quad (1.25)$$

If the integration is over a compact domain D bounded by a vorticity surface ∂D such that the normal to the surface vorticity component is zero, $\omega_n|_{\partial D} = 0$, in particular over infinite flow domain, the surface integral vanishes and consequently helicity is the exact invariant of the Euler equations, as energy is. Helicity is a pseudo-scalar and can be positive or negative.¹⁶ The same is true for the viscous term in Eq. (1.24). In difference to the energy and vorticity balance equations the viscosity can be either a sink of helicity or a source.

¹⁶Solid boundaries at which of course $\boldsymbol{\omega} \cdot \mathbf{n}|_{\partial \mathbf{D}}$ are the helicity sink/source. At the solid boundaries the inviscid flows generally form the tangential discontinuities in the velocity field and therefore the vorticity sheets.

Helicity is non-zero only for the flows lacking the reflectional (mirror) symmetry and as was mentioned above is a topological invariant.¹⁷ Briefly helicity is usually interpreted as a measure of knottedness of divergence free solenoidal vector field lines, and in this case the vortex lines: helicity is proportional to the number of knots and linkages made by the vortex lines. The helicity then would be a discrete quantity akin to topological Hopf invariant. In fact a more general interpretation of helicity as a continuous topological invariant measure was analyzed by Arnold. The vortex lines in a compact fluid domain can be closed, knotted or not, or end at the boundaries. But they may have a non-trivial topology, e.g., winding about surfaces of arbitrary complexity (for non-differentiable velocity flow field), or ergodically filling a fluid sub-domain without end.^k In the ergodic configurations the vortex lines can form asymptotically close linkages and knots and in general possess arbitrarily complex topology that remain topologically invariant under smooth (but not necessarily invertible) mappings induced by fluid motion.

Helicity is a particular topological invariant of flows of inviscid fluids. However helicity is not the only invariant defining the topology of divergence free vector fields. It is easy to construct a vector field configuration that has zero helicity but is topologically complex and knotted, a usually demonstrated simple example being the Boromean ring, e.g., Moffatt (1969). In general there exists an infinite family of topological invariants for frozen-in fields more nonlinear than helicity as was shown in Tur and Yanovsky (1993). They are all the consequences of the Kelvin's theorem of conservation of circulation in inviscid fluids. It is however the connection between the helicity density and the nonlinear term in the Navier-Stokes equation that makes suggestive the role that helicity may play in turbulence: maximal helicity in a compact flow domain or sub-domain is given by Beltrami flows yielding zero to the nonlinear coupling term in the Navier-Stokes equations.

Note that the viscous term in the balance equation (1.24) is not necessarily a dissipative one, meaning that it does not necessarily decrease $|H|$. The viscosity can reduce the absolute value helicity or generate it. As helicity is not positively defined so $\nu\omega \cdot \mathbf{curl}\omega$ is

¹⁷Analogous to helicity invariant for Eq.(1.13) is cross helicity $A = \int \mathbf{v} \cdot \mathbf{B}d\mathbf{V}$.

not. Generally there is no need for external helical forces to create a non-zero helicity in a turbulent flow. Although quite obvious this property results in surprising consequences for turbulence structure.

2 Kolmogorov Theory of Homogeneous Isotropic Turbulence (K41): Part 1

The Kolmogorov theory of homogeneous isotropic turbulence-HIT- has been dominant for over half a century among the scientists seeking fundamental understanding of turbulence. The Kolmogorov theory is an important subject of geophysical studies. Many geophysicists view turbulence from a fundamental view point may be because they are best exposed to its many grandiose manifestations. On the other hand practitioners of turbulence in civil engineering and aeronautics who are mainly concerned with turbulence near to the solid boundaries, highly anisotropic boundary layer (BL) turbulence, usually have very limited use for the Kolomogorov theory, some of them viewing it as may be correct but for a highly idealized model that is not relevant for real life applications. Turbulent BL flows are full of structures. Although these structures were not given real mathematical description they are clearly visualized, so that the engineers are convinced that they are all important in their contribution to the physical parameters important for applications, e.g., heat and mass transfer, turbulent friction (drag) at the boundaries, etc. There are no such structures in the Kolmogorov model of turbulence. Thereby follows the usual and liely false argument that CS is the intrinsic feature of specifically BL turbulence.

On the other hand the celebrated prediction of Kolmpogorov theory, the $-5/3$ power law for the energy spectrum is observed in laboratory experiments modeling HIT, e.g., decaying turbulence past a grid and in part in geophysical phenomena, albeit with modifications and degree of uncertainty.^l This law is to some extent confirmed in DNS, the Reynolds numbers are not high enough for confidence. The most surprising thing about this spectrum is that it is observed even in conditions remote from HIT such as for instance in large scale and highly anisotropic atmospheric flows.

The Kolmogorov theory has been so dominant and for such a long time among so many thinking deeply on the nature of turbulence that

it has become almost an Aristotelian must. It is very difficult to make even a slightest dent in its basic postulates. Because of this and for immediate reference by readers it is useful to remind here briefly the main tenets of the theory even though it has been done in thousands of papers before in dozens of different ways. From the outset it is necessary to state that although some of the basic principles of Kolmogorov theory will be asserted wrong here the Kolmogorov energy spectrum will remain intact. However the physics behind the spectrum is quite different from the one postulated in the Kolmogorov theory or implied by it. The Kolmogorov theory is often portrayed as a simple scaling relation for the flow energy spectrum, i.e., the distribution of energy of turbulent motion among different scales of motion, the velocity harmonics. But in actual fact the theory is very complex logically, although the mathematics is indeed simple. It relies on deep intuition into the nature of the Navier-Stokes equations which while cannot be solved still can be analyzed to some, admittedly limited, degree and furnish indications to at least certain features and properties of turbulent motion. Such analysis of the Navier-Stokes equation had been carried out by generations of scientists and engineers for over a half century prior to Kolmogorov contribution and surely played role in the formation of the Kolmogorov theory of 1941.

The Kolmogorov basic concept of 1941, often called K41, starts from the preceding poetic observation of Richardson made quite earlier in 1922 that rhymes ostensibly like this:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity (In the molecular sense)¹⁸

Laminar flows are driven by extraneous forces, like pressure gradient in pipe flows. The mechanism for the laminar flow to become turbulent is, as was emphasized above, a distinct scientific problem beyond the scope of this paper. The fact is that laminar flows are almost always unstable. The simplest way to obtain turbulence is to stir fluid in a semi- random way. Imagine a paddle stirring fluid in

¹⁸This rhyme is quoted and misquoted by many. It carries such clear and simple picture of hierarchy of whorls that it is impossible to resist the temptation to quote it once again

a large pail. Obviously this paddle can create a whirl or a vortex of the size comparable or bigger than the size of the paddle. This prime vortex size will be denoted as the integral scale L . So far the motion is laminar. But the prime vortex or vortices are totally unstable and decompose into smaller vortices customary called eddies. And these eddies will decompose into smaller eddies and then into even smaller eddies and so forth till such time when eddies become small enough so that the viscous term in the Navier-Stokes equations becomes dominant, it is recalled that it is the second order space derivative of the velocity field, and these smallest viscous scale $l_d \ll L$ eddies dissipate their motion into heat. Obviously the number of l_d scale eddies is much larger than the integral scale L eddies, since the total incompressible fluid volume remains constant. The whole process can be likened to a cascade of water through the many steps in some waterfalls before sinking in a water reservoir below. But in this case the energy cascades through the eddies not in physical space but in the space of scales filling the same fluid domain.

The external forces are opposed by the internal viscosity friction of fluids and if there are boundaries by their friction at the boundaries. In order for a steady state to set it should be that the same amount of energy as is injected into fluid by the stirring paddle to generate integral scale eddies is dissipated into heat by the many more small viscous scale eddies: the viscous forces balance the external forces so that the work done by the external forces is equal to the energy dissipated by the viscous forces.

The basic postulate of Kolmogorov theory is that the instabilities of integral scale laminar flow give rise to progressively smaller scales motion. As in the picture of Richardson the big eddy generates progressively smaller and smaller eddies. The reason and possibility for eddies splitting into ever smaller eddies lies in the nonlinear nature of the Navier-Stokes equations for the velocity field. The simple mathematical details are discussed below in the next Section, but it is quite clear that when the Eqs. (1.1) - (1.3) are rewritten in Fourier space for the velocity field harmonics $\mathbf{v}(\mathbf{k}, \mathbf{t})$ as a function of wave vectors \mathbf{k} :

$$\mathbf{v}(\mathbf{k}, \mathbf{t}) = (2\pi)^{-1} \int \mathbf{v}(\mathbf{r}, \mathbf{t}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{V}. \quad (2.1)$$

The quadratic nonlinear term couples triads of velocity harmon-

ics, $\mathbf{v}(\mathbf{k})$, $\mathbf{v}(\mathbf{q})$ and $\mathbf{v}(\mathbf{k} - \mathbf{q})$ or the triads of eddies in the space of scales and therefore generates in general different triadic scales of fluid motion. But still it is a non-trivial assumption in the spirit of Richardson that the generated scales of the velocity harmonics are predominantly smaller than the prime characteristic scale of fluid motion that is called the integral scale L . The integral scale can be associated with an external body force acting on the flow, or with a leading scale of natural instabilities of the incipient laminar flow, e.g., caused by interaction with boundaries. But in principle the non-linear interactions could have created the opposite process to cascade, the growth of eddies, or what is usual to call the inverse cascade. However, most probably as a rule the inverse cascade does not realize in $3D$ world.¹⁹

The next fundamental postulate of Kolmogorov is that in the limit of high Re and after many cascade steps the small scale eddies of turbulence are universal, isotropic and homogeneous. That is to say that a turbulent motion that ensues from the initial integral scale motion, at the scales $l \ll L$ become self similar and universal. Therefore it

¹⁹In $2D$ world on the contrary the inverse cascade would be a reality and eddies would predominantly grow in size rather than become smaller (e.g., Kraichman, 1962). The expectations of many including myself had been that the inverse cascade in one way or another would be realized in the flows like large scale atmospheric flows where the horizontal scales of motion are very large by comparison with the vertical depth of Earth atmosphere; the latter is limited by gravity. The inverse cascade would have been an easy and very tempting mechanism for formation of large scale atmospheric structures. These expectations are almost certain to be unfulfilled. The flows anisotropy does not make turbulence $2D$ and the inverse cascade does not occur. On the contrary, most probably the general properties of the turbuleny flows remain very much the same despite the strong disparity between the scales of fluid motions in different dimensions. If inverse cascade realizes it is only as some sort of instabilities, or fluctuation in the otherwise unidirectional flow of energy from large scales to small ones. In this one might invoke a philosophical wisdom, akin to the second law of thermodynamics in equilibrium systems of which turbulence is not the one to be sure, that the real world all non - equilibrium systems tend to have a dominant propensity for decomposition and dissipation. And that the growth in size and extent of organization as is sometimes observed is a temporal fluctuation that should be strongly driven by appropriately arranged external forces. But this may be wrong or incomplete wisdom since we know that in turbulence the dissipation is possible only if **BCC** are formed. In anisotropic flows **BCC** may stretch indefinitely in the flow direction since there are no apparent scales limiting the size of **BCC** in unbounded direction.

should be independent of the prime flow integral scale L and of Re . The independence of L and of Re signifies that the motion at these scales is totally dominated by the local in wavenumbers nonlinear couplings. Let us comment that the assumption that only local non-linear coupling remains would indicate that somehow the non-linear term in Eqs. (1.1) - (1.3), which generally describes all the interactions would reduce in some rigorous mathematical analysis of the Navier-Stokes equation, if such was possible. Some of the interactions, the non-local ones, would cancel out and only the harmonics with the local triads, such that all the three wavenumbers $|\mathbf{k}|$, $|\mathbf{q}|$ and $|\mathbf{k} - \mathbf{q}|$ are the same order of magnitude, would remain. This is an important postulate of locality of interactions in Fourier space to bear in mind.

On the other hand by the nature of Eq. (1.1) there are some relatively small viscous scales of order of l_d at which the linear viscous term in Eq. (1.1), since it is the second order velocity field space derivative, would necessarily become dominant for any given arbitrary large value of Re . When eddies become so small the viscous friction just dissipates their motion into heat.²⁰ Also, the universality naturally implies isotropy and homogeneity of small scale eddies in the inertial range after many steps of cascade, since there is no singled out scale in the inertial range to associate with either anisotropy, or inhomogeneity. This is the essence of the Kolmogorov HIT model. For consistency let us remind briefly the statistical mathematical frame of this model.

The values of all turbulent fields are extremely variable in space and time and therefore it is natural to consider $\mathbf{v}(\mathbf{r}, \mathbf{t})$ as a random function superimposed on the mean flow velocity, including the case of zero mean velocity. If the turbulent velocity were, indeed, a random field it would be fully characterized by the probability density:

$$\begin{aligned}
 W\{v\} D\mathbf{v} &= \mathbf{W}\{\mathbf{v}^{(N)}\} \mathbf{d}^{(N)} \mathbf{v}^{(N)} = \\
 &= \lim_{N \rightarrow \infty} W\{\mathbf{v}(\mathbf{r}_1, \mathbf{t}_1), \mathbf{v}(\mathbf{r}_2, \mathbf{t}_2), \dots, \mathbf{v}(\mathbf{r}_N, \mathbf{t}_N)\} \mathbf{d}\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{d}\mathbf{v}_N,
 \end{aligned}
 \tag{2.2}$$

²⁰The energy spectrum is usually assumed to be exponentially decaying in the viscous subrange of wavenumbers (Monin and Yaglom, 1975). The situation may be more complex than this in reality

$$\int W\{\mathbf{v}\} \mathbf{D}\mathbf{v} = 1,$$

where the functional $W\{\mathbf{v}\}$ gives the probability of having certain velocity field space/time $4D = (3 + 1)$ realization, at all the points in space and at all times, in the multidimensional functional space of all possible realizations.

If homogeneity and steady state of the velocity field are assumed, as befits HIT model, then the turbulent flow can be described by means of the velocity correlation functions of different orders at all points and all times, i.e., by all correlation functions of the type:

$$\begin{aligned} \langle \mathbf{v}(\mathbf{r}_1, \mathbf{t}_1), \dots, \mathbf{v}(\mathbf{r}_N, \mathbf{t}_N) \rangle &= \langle \mathbf{v}(\mathbf{0}, \mathbf{0}), \dots, \mathbf{v}(\mathbf{r}_{N-1}, \mathbf{t}_{N-1}) \rangle = \\ &= \int \prod_{i=1}^{N,M} v(r_i, t_i) W\{v\} Dv. \end{aligned} \quad (2.3)$$

In principle the knowledge of all the $1, 2, \dots, N, 1, 2, \dots, M$ space/time correlation functions, if they exist and are finite in the limit of $N, M \rightarrow \infty$ would be equivalent to the full knowledge of $\mathbf{v}(\mathbf{r}, \mathbf{t})$.

In real experiments the ergodic assumption is applied and the ensemble averaging is understood in reduced sense as the time averaging over the values of $\mathbf{v}(\mathbf{r}, \mathbf{t})$ at a point. This is because practically all measurements can be made only at one, or at best few points in order to calculate the velocity derivatives, in a fluid flow. In numerical simulations even further approximation is usually accepted. If the space-time lattice on which the discretized $\mathbf{v}(\mathbf{r}, \mathbf{t})$ is calculated is dense enough, than the space and time averaging over the grid points is assumed to be sufficient for accurate calculations of the real ensemble averaged correlation functions. Even if one space realization of turbulence is represented by large enough N it is enough to calculate the space average for certain quantities. This is what is done usually in DNS for the models of decaying turbulence. In this model turbulence is triggered at t_0 by injection of energy into flow and then allowed to evolve in time as it follows the Navier-Stokes dynamics. In real DNS even now the number of grid points is still too limited for calculating high order correlation functions. Moreover since some of the basic quantities, enstrophy and its generation rate for instance, are determined by relatively small subsets of grid points.

The experiment in decaying turbulence shows that for one space point, one time velocity $\mathbf{v}(\mathbf{r}, \mathbf{t})$, the probability functional $W\{\mathbf{v}\} \mathbf{D}\mathbf{v}$

reduces to a nearly Gaussian distribution function. But $W\{\mathbf{v}\}D\mathbf{v}$ is a strongly non-Gaussian functional for space-time variations of velocity at two or more points or its space derivatives related quantities, such as stress tensor e_{ik} , vorticity ω , etc. This deviation from the normal Gaussian law is associated with the phenomenon of intermittency.²¹ But let us go back to the Kolmogorov theory that does not concern itself with intermittency.

The velocity of the largest eddies v_L is comparable to the total variation of velocity Δv_L separated by the distance L that is the scale of the whole flow. Actually v_L is smaller by a factor of two or three but it does not matter really because if $Re = \Delta V_L L/v \gg 1$ for the whole flow it follows that:

$$v_0 l_0 \gg 1, \tag{2.4}$$

where l_0 is the size of the biggest eddies and by $v_0 \approx v_L$ we rather understand the root mean square of the fluctuating velocity typical; for the ensemble of proto-eddies, that is $\sqrt{v_0^2}$.

The pro-eddies are the energy containing in the sense that they hold the major part of the energy associated with the total fluctuating part of the velocity, i.e., $E \approx v_0^2$. Successively the smaller eddies are characterized by successively smaller, but still large Reynolds numbers and contain less energy,

$$Re \gg Re_i = v_{(i)} l_i / v > Re_{i+1} \gg 1, \tag{2.5}$$

for each generation of eddies i . To avoid confusion it should be stressed that the notion of eddies is not really well defined. It would be safe to see the velocities $v_{(i)}$ as a characteristic fluctuating velocity variance over a distance l_i , rather than the velocities of well delineated vortices. In fact the latter can be seen as the definition of eddies. Eventually the size and the corresponding eddies velocity become small enough so that:

$$Re_d = v_d l_d / v \approx 1, \tag{2.6}$$

So that at these scales the dissipative term in the Navier-Stokes equations becomes of the order as the nonlinear coupling. The scales of

²¹in Laboratory conditions the decaying turbulence is realized in turbulent flows past a grid described in Endnote **g** in the end of the text.

order of l_d are called the *dissipation range*, while the scales from the interval:

$$L \approx l_0 \gg l_d, \tag{2.7}$$

this range is called the *inertial range*. For each scale l_i let us introduce the time scale:

$$\Delta t_i = l_i/v(i), \tag{2.8}$$

during which the eddy is likely to decompose into smaller ones. The characteristic time $\Delta t_0 \gg \Delta t_i$ is called (large) *eddy turnover time*. During the time comparable with the eddy turnover time the proto-eddies will decompose enough, as the result of nonlinear interactions, so that the inertial range is formed and the dissipation range is reached.²²

It should be taken into account that in real conditions there is always present a mean inhomogeneous flow with velocity $|\mathbf{U}(\mathbf{r}, \mathbf{t})| > \Delta \mathbf{v}_L$. It is quite clear that for the assumption of homogeneity and isotropy of eddies (in the statistical sense) it is necessary that the following condition is fulfilled:

$$|\partial_k U_i|^{-1} \gg \Delta t_0 \gg \Delta t_i, \tag{2.9}$$

so that it is clear that the assumptions of HIT can be understood only locally in globally inhomogeneous flow and are applicable only to eddies from the inertial range.

Assume that turbulence in the inertial range is in a steady state. It is a safe assumption provided that the considered time interval satisfies (2.9). Then it should be from the balance of energy requirement that the average energy flux *per unit volume per sec* passing through the successive generations of eddies with velocities v_i remains constant. Or more correctly the ensemble averaged energy flux density $\langle \epsilon \rangle = \text{const}$. Clearly the flux is in the space of scales l_i diminishing with the growth of generations of eddies i , or in the Fourier space of wavenumbers $k_i \approx l_i$. In physical space there is no systematic ensemble averaged flux in HIT. Since the dimension of $\langle \epsilon \rangle$ is $\text{length}^2 \times \text{time}^{-3}$ (it is reminded that the density $\rho = 1$) from purely dimensional considerations it follows:

$$\langle \epsilon \rangle \approx \Delta v_L^3/l_0 \approx v_0^3/l_0 = \dots = v_{(i)}^3/l_i. \tag{2.10}$$

²²In DNS of decaying turbulence it is usually safe to wait for a few Δt_0 to achieve this state of developed turbulence.

Let us introduce a very useful quantity called eddy viscosity:²³

$$v_{eddy} \approx \Delta v_L l_0 \approx v_0 l_0; v_{i,eddy} = v_{(i)} l_i. \quad (2.11)$$

Then it follows:

$$\langle \epsilon \rangle \approx v_{eddy} (v_{(i)} / l_i)^2 = v (v_d / l_d)^2 \quad (2.12)$$

Note that the eddy viscosity is scale dependent and in general class of flows space-time dependent. The eddy viscosity can be also introduced as follows:

$$\langle \epsilon \rangle \approx \langle \epsilon(l_0) \rangle \approx -v_{eddy} \langle \{ \partial_k \tilde{v}_i(l_0) + \partial_i \tilde{v}_k(l_0) \}^2 \rangle \approx -2v \langle e_{ik}^2 \rangle, \quad (2.13)$$

where $\tilde{v}_i(l_0)$ and $\langle \tilde{\epsilon}(l_0) \rangle$ are the smoothed over the small scales mean velocity field and dissipation rate respectively that are now dependent on the large scale $-l_0$ structure only and the differentiation is done accordingly over these large scales. What the relation (2.13) means is a simple statement that as much energy is passed over from large scale eddies to the smaller ones caused by the stress at the large scales will be eventually dissipated by the molecular viscosity.

Now we can find again from purely dimensional considerations the eddy velocity as a function of their size. Since neither Re nor L can enter by the assumptions of HIT it follows unambiguously:

$$v_{(i)} = \langle \epsilon \rangle^{1/3} l_i^{1/3}, \quad (2.14)$$

providing an explicit law for the distribution of velocity fluctuations in turbulence as a function of their scale. The definition of v_i (2.14) makes the relation (2.12) an identity. The corresponding characteristic Kolmogorov eddy number i turnover time is:

$$\Delta t_i \approx l_i / v_{(i)} = \langle \epsilon \rangle^{-1/3} l_i^{2/3}. \quad (2.15)$$

A very important relation connects L and l_d as follows:

$$L / l_d \approx (v_0 l_0 / v)^{3/4} \equiv Re^{3/4}. \quad (2.16)$$

²³The eddy viscosity is called like this because if the eddy size and velocity l_i, v_i are substituted by the molecular free path and velocity λ_{free}, v_{mol} one obtains the definition of molecular viscosity ν .

The relation (2.16) furnishes self consistency to the definition of the inertial range, through Re and this is what it should be because Re is the only intrinsic parameter in the Navier-Stokes equation.

The Eqs. (2.14) - (2.16) constitute the essential basis of the Kolmogorov theory. It is amazingly simple and allows the calculation of most quantities that are needed for practical applications (Monin and Yaglom). Rarely in the history of science were such profound conclusions made on the basis of so few assumptions, which seem all quite innocuous and even obvious, and with such little mathematical complexity. The generality of the results is also puzzling, since they are universally applicable for all turbulent flows, in pipes, atmosphere, geophysics, oceanography, wherever one is concerned with flows far enough from the boundaries.²⁴

It is practically more convenient, although not really more fundamental to consider the Kolmogorov theory in Fourier space. Let us introduce the energy spectrum as follows:

$$E(k) = k^2 \int E(\mathbf{k}) d\Omega = (4\pi)k^2(2\pi)^{-3}(1/2) \int \langle \mathbf{v}(\mathbf{0}) \cdot \mathbf{v}(\mathbf{r}) e^{-i\mathbf{k}\mathbf{r}} d\mathbf{V} \rangle \\ \equiv (1/\pi)k^2 \int \langle \mathbf{v}(\mathbf{0}) \cdot \mathbf{v}(\mathbf{r}) \rangle (\sin kr)/krr^2 d\mathbf{r}, \quad (2.17)$$

where $d\Omega$ is the solid angle differential. The average energy density is now:

$$\langle E \rangle = \int E(k) dk = 1/2 \langle v_i^2(0,0) \rangle. \quad (2.18)$$

Now one applies to the spectrum $E(k)$ the Kolmogorov assumptions in a slightly different manner. From the general dimensional considerations it follows:

$$E(k) = A \langle \epsilon \rangle^{2/3} k^{-5/3} \phi_1(Re\phi_2(kL)). \quad (2.19)$$

In the limit $Re \rightarrow \infty$ we can expect universality and Re independence. This fixes $\phi_1 = 1$. The postulated independence of the energy spectrum from the large scale motions fixes $\phi_2 = 1$. So the Kolmogorov spectrum that will be called also the K41 spectrum in what follows becomes:

$$E(k) = A \langle \epsilon \rangle^{2/3} k^{-5/3}, \quad (2.20)$$

²⁴The distance from the boundaries is meant in dimensionless units defined by Re . In physical units this can be very small for high values of Re .

while:

$$k_0 \ll k \ll k_d = l_d^{-1}. \quad (2.21)$$

The spectrum (2.20) in Fourier space is essentially equivalent to the law (2.14) in physical space.²⁵ It should be noted that in the original Kolmogorov theory $\langle \epsilon \rangle$ was rather identified with the average rate of energy viscous dissipation. But the average energy flux is of course equal to the average rate of viscous dissipation.

Fundamental question, in particular for DNS of turbulence, is the number of independent velocity harmonics, or the number of degrees of freedom that are necessary for adequate representation of turbulence. This question was answered in the classical analysis of Landau and Lifshitz (1987) in the following manner. Let us assume that m is the number of modes that can be densely packed into unit volume of turbulent flow. This number from dimensional considerations can be a function of only the time independent parameters $\langle \epsilon \rangle$ and v . However there is only one quantity with the dimension of length that can be constructed with these two parameters and that is the Kolmogorov length scale l_d defined by (2.16), or equivalently as $l_d \approx (\langle \epsilon \rangle / v^3)^{-1/4}$. Since m has the dimension of $length^{-3}$ it follows that $m = l_d^{-3} \approx (\langle \epsilon \rangle / v^3)^{3/4}$. The total number of modes in the volume $L^3 \approx l_d^3$ is then:

$$N_{modes} \approx (L/l_d)^3 = (k_d/k_0)^3 \approx Re^{9/4}. \quad (2.22)$$

Since the Reynolds number is typically very big the total number of independent modes, as it follows from K41 is huge. This in particular explains why it is so difficult to simulate turbulence and why it is so difficult to have good meteorological models.²⁶ Another extremely

²⁵Kolmogorov formulated his theory in physical space of scales. Obukhov reformulated it in Fourier space. So in truth the spectrum should really be called Kolmogorov - Obukhov spectrum. For brevity we shall refer to the energy spectrum (2.21) in the manner often accepted in literature as K41 spectrum and when considered together with all the underlying assumptions as K41 theory.

²⁶Because for the typical values Re in geophysical conditions no computer power will be ever sufficient in accordance with (2.22). Hence the meteorological modeling is based on assumptions as regards the high wave numbers $k \leq k_d$ flow properties of the flow so that to take into account of them in an averaged way as some turbulent viscosity (see Section 3) and cut down on the number of participating velocity harmonics. The shortcoming is that with no proper understanding of the small scale nature of turbulent flows such approximation can be way off the reality.

important conclusion from K41 is that enstrophy is all defined by the smallest scales of order l_d . Indeed, it is easy to see from the general definition of enstrophy $\Omega = \omega^2$ that $\omega_{(i)} \approx l_i v_{(i)} \approx L_i^{-2/3}$. The smaller is the eddies scale the more vorticity they have. More formally we introduce the enstrophy spectrum as:

$$\begin{aligned} \langle \Lambda/2 \rangle &= \langle \omega^2/2 \rangle = k^2/2 \int \Lambda(\mathbf{k}) d\Omega = \\ &= (2\pi)k^2(2\pi)^{-3} \int \langle \omega(\mathbf{0}) \cdot \omega(\mathbf{r}) \rangle e^{-i\mathbf{k}\mathbf{r}} d\mathbf{V} = \int E(k)k^2 dk \approx \\ &\approx \langle \epsilon \rangle^{2/3} k_d^{4/3} \approx \langle \epsilon \rangle^{2/3} l_d^{-4/3} = O(Re), \end{aligned} \quad (2.23)$$

where we used the relation (2.16). The Eq. (2.23) is absolutely fundamental for understanding turbulence. It shows that Re as grows so does enstrophy and this growth is entirely due to the smallest scales or eddies in the inertial range, or the largest wavenumbers in the inertial range. The wavenumbers in the dissipation region $k > k_d$ are fast, ostensibly exponentially decreasing and hence do not influence much the average enstrophy value.

On the contrary the average energy is defined almost entirely by the largest scale eddies. Indeed it follows from (2.18) and K41 that:

$$\langle E \rangle \approx \langle \epsilon \rangle^{2/3} L^{2/3} \approx \langle \epsilon \rangle^{2/3} l_0^{2/3} = \langle \epsilon \rangle^{2/3} k_0^{-2/3}. \quad (2.24)$$

This is a very beautiful picture that should be deeply felt for good understanding of what happens in turbulence: energy from the energy containing large eddies flows steadily through many generations of successively smaller eddies, increasing the enstrophy in each successive generation of eddies while maintaining their energy negligible by comparison with the prime eddies and eventually dissipating via the smallest eddies through the molecular viscosity into heat. This is called the energy cascade since it literally reminds the cascading waterfalls. But water cascades in physical space and energy in the space of scales or wavenumbers, remaining in the same place in physical space (unless carried away by the mean flow).

With the understanding of the principles of the Kolmogorov theory it is necessary to return back and inspect with care the exact Eq.(1.22). It must become clear now that the cascade seems the only

way in which the Navier-Stokes equations can allow the energy dissipation. It can happen only through the growth of enstrophy. And since $Re \rightarrow \infty$, which is the same limit of course as $\nu \rightarrow \infty$, the finite rate of dissipation equal to the energy flux in the space of scales $< \epsilon >$ can be only attained if enstrophy tends to infinity in the same limit. But the growth of enstrophy is possible only through the stretching of vortex lines by the fluid motion which in conjunction with the erratic nature of this fluid motion inevitably generates progressively smaller scales or progressively smaller eddies. This is the gist of the cascade mechanism that had been first envisioned by Richardson and brought to the final extraordinary conclusions in K41. And this was done with almost no mathematical tools employed beyond the scaling analysis and elementary derivations.

3 Kolmogorov Theory of Homogeneous Isotropic Turbulence (K41): Part 2

For clear understanding it is necessary to furnish formal definitions of certain basic quantities and relations that will be used in what follows as a rule without referring to the original works. The velocity field is assumed to be a random function, statistically homogeneous, isotropic and stationary function. What it means is that the correlation functions made of the velocity field are invariant with regard to translations and rotations, but generally not invariant with regard to reflection. With these assumptions the most general expression for the second rank velocity correlation tensor is:

$$B_{ij}(r, t) = \langle v_i(r, t)v_j(0, 0) \rangle = A(r, t)\delta_{ij} + B(r, t)x_ix_j/r^2 + C(r, t)\epsilon_{ijl}x_l. \quad (3.1)$$

The lack of mirror symmetry yields the third term in the r.h.s. of (3.1). From the incompressibility condition (1.3) it follows that:

$$\partial_j B_{ij} = 0, \quad (3.2)$$

so that:

$$\partial A/\partial r + \partial B/\partial r - B/r = 0. \quad (3.3)$$

Let us also assume that the velocity field is a stationary random function. Then its Fourier transform in the space of wavenumbers

and frequencies, (k, f) - space is a generalization of (2.1) as follows:

$$\mathbf{v}(\mathbf{k}, f) = (2\pi)^{-4} \int \mathbf{v}(\mathbf{r}, t) e^{-i\mathbf{k}\mathbf{r} - ft} dV dt, \quad (3.4)$$

subject to the finite norm condition:

$$\|\mathbf{v}(k^{(4)})\|^2 = \int |\mathbf{v}(k^{(4)})|^2 dk^{(4)} < \infty, \quad (3.5)$$

and similarly in physical space. We introduced a formal a notation $k^{(4)} = (\mathbf{k}, f)$. Since $\mathbf{v}(\mathbf{r}, t)$ is real it follows for the conjugate:

$$\mathbf{v}(k^{(4)})^* = \mathbf{v}(-k^{(4)}). \quad (3.6)$$

Using the Fourier transformation of the velocity field we introduce the spectral tensor:

$$\langle v_i(k^{(4)}) v_j(k^{(4)}) \rangle = F_{ij}(k^{(4)}) \delta(k^{(4)} + k'^{(4)}). \quad (3.7)$$

Hence comparing with (3.1):

$$F_{ij}(k^{(4)}) = (2\pi)^{-4} \int B_{ij}(\mathbf{r}, t) d\mathbf{r} dt. \quad (3.8)$$

Respecting translational and rotational invariance as in (3.1) - (3.3) yields the most general form of the second rank tensor as follows:

$$F_{ij}(k^{(4)}) = A(k, f) \delta_{ij} + B(k, f) k_i k_j + C(k, f) \epsilon_{ijl} k_l, \quad (3.9)$$

where the wave number $k = |\mathbf{k}| = \sqrt{k_i^2}$. The continuity condition (1.3) in Fourier space yields:

$$k_i v_i(k^{(4)}) = 0; k_i F_{ij} = 0. \quad (3.10)$$

Then the most general expression for the second rank spectral tensor can be rewritten as follows:

$$F_{ij}(k^{(4)}) = E(k, f) / 4\pi k^2 [\delta_{ij} + k_i k_j / k^2] + i \epsilon_{ijl} k_l H(k, f) / 8\pi k^4. \quad (3.11)$$

The scalar functions $E(k, f)$ and $H(k, f)$ are respectively the energy and helicity spectra. Let us show this. Integrating the symmetric

part F_{ij} over the wavenumbers and frequencies and together with (3.8) obtain for the average energy density:²⁷

$$\langle E \rangle = 1/2 \int F_{ij}(k^{(4)}) dk^{(4)} = \int E(k, f) dkdf = 1/2 \langle v_i^2(0, 0) \rangle, \quad (3.12)$$

where $dk^{(4)} = d\mathbf{k}df$. The one time wave number spectrum for the steady state turbulence:

$$E(k) = \int E(k, f) df = (2\pi)k^2 \int \langle \mathbf{v}(0, 0) \cdot \mathbf{v}(\mathbf{r}, 0) \rangle (\sin kr/kr) dr. \quad (3.13)$$

Note that for $E(k) = A \langle \epsilon \rangle^{2/3} k^{-5/3}$ the velocity correlation function in the inertial range is:

$$\langle \mathbf{v}(0) \cdot \mathbf{v}(\mathbf{r}) \rangle \propto \langle \epsilon \rangle^{2/3} L^{2/3} (1 - A_{cor} r^{2/3} / L^{2/3});$$

$$l_d \ll r \ll L; A_{cor} = const \approx 1. \quad (3.14)$$

The antisymmetric part of $F_{ij}(k^{(4)})$ requires a bit more of inspection. From the definition of vorticity it follows in Fourier space:

$$\omega_i(k^{(4)}) = i\epsilon_{ijl}k_l v_j(k^{(4)}), \omega(k^{(4)}) = i[\mathbf{k} \times \mathbf{v}(k^{(4)})]. \quad (3.15)$$

Consider now the scalar product:

$$H(k^{(4)}) = \mathbf{v}(k^{(4)}) \cdot \omega(-k^{(4)}). \quad (3.16)$$

Only the real part of $H(k^{(4)})$ is non-zero. Together with the definition (3.15) it follows for the helicity spectral density:

$$H(k^{(4)}) = 2\mathbf{k} \cdot [\text{Re}\mathbf{v}(k^{(4)}) \times \text{Im}\mathbf{v}(k^{(4)})].^{28} \quad (3.17)$$

It is useful to notice that in the same way it follows for the energy spectral density:

$$E(k^{(4)}) = 1/2\{[\text{Re}\mathbf{v}(k^{(4)})]^2 + [\text{Im}\mathbf{v}(k^{(4)})]^2\}. \quad (3.18)$$

²⁷Since the statistical description of turbulent fields is considered the total energy is substituted by the ensemble averaged energy and the total helicity with ensemble averaged helicity. Additionally, we are considering the statistically homogeneous steady state flow, so that it follows $\langle v(r, t)^2 \rangle = \langle v(0, 0)^2 \rangle$.

²⁸The real value Re should not be confused with the Reynolds number *Re*.

Using Parcival's theorem and averaging over the ensemble we obtain after some identical transformations:

$$\begin{aligned} \langle h \rangle &= \lim_{V,T \rightarrow \infty} (VT)^{-1} H = \lim_{V,T \rightarrow \infty} \int h dV dt = \\ &= \lim_{V,T \rightarrow \infty} \int \mathbf{v} \cdot \boldsymbol{\omega} dV dt = \int \langle H(k^{(4)}) \rangle dk^{(4)} = \\ &= (2\pi)^{-4} \int \langle \mathbf{v}(k^{(4)}) \cdot \boldsymbol{\omega}(-k^{(4)}) \rangle dk^{(4)} = \int H(k, f) dk df = \int H(k) dk. \end{aligned} \tag{3.19}$$

It is not difficult to get convinced that $H(k, f)$ is the same as in (3.11). In other words $H(k, f)$ is the helicity spectrum like $E(k, f)$ is the energy spectrum. We will be interested mainly in the wave number dependence of the helicity spectrum Hk . It can be derived alternatively as follows.

Let us introduce the one time helicity correlation function:

$$h(r) = \langle v_i(0)\omega_i \rangle. \tag{3.20}$$

Then it follows identically:

$$\langle h(0) \rangle = \langle h \rangle. \tag{3.21}$$

Also:

$$\begin{aligned} H(k) &= k^2 \int H \mathbf{k} d\Omega = (4\pi)/(2\pi)^3 \int h(r) e^{-ikr} dV = \\ &= (4\pi)k^2/(2\pi)^3 \int \langle \mathbf{v}(0) \cdot \boldsymbol{\omega}(r) \rangle (sinker/kr) r^2 dr, \end{aligned} \tag{3.22}$$

where $d\Omega$ is the solid angle differential element. Hence the average helicity density is:

$$\langle h \rangle = \int H(k) dk \equiv \langle v_i(0)\omega_i(0) \rangle. \tag{3.23}$$

Let us reformulate the balance equations (1.16) and (1.24) for the averaged quantities $\langle E \rangle$ and $\langle h \rangle$. Since the fluxes in physical space are zero for HIT the ensemble averaging of the balance equations yields:

$$\partial_t \langle E \rangle = \langle v_i F_i \rangle = -2\nu \int k^2 \langle E(k) \rangle dk. \tag{3.24}$$

Evidently:

$$-\langle \epsilon \rangle = -2\nu \langle e_{ik}^2 \rangle = -\nu \langle \omega^2 \rangle = -2\nu \int k^2 E(k) dk = O(1), \quad (3.25)$$

where Re independent average energy flux in Fourier space of wave-numbers $\langle \epsilon \rangle$ is the same as the one that stands in the Kolmogorov spectrum. Similarly instead of (1.24) it follows:

$$-\langle \epsilon_h \rangle = \partial_t \langle h \rangle = -2 \langle \omega \cdot \text{curl} \omega \rangle = -2\nu \int k^2 H(k) dk = O(1), \quad (3.26)$$

where $\langle \epsilon_h \rangle$ is the average helicity flux in Fourier space. And at last for the enstrophy balance Eq. (1.20):

$$\partial_t \langle \Lambda \rangle = \langle \omega \cdot (\omega \cdot \nabla) \omega \rangle = \nu \langle \text{curl} \omega \rangle^2 \quad (3.27)$$

where:

$$-\nu \langle (\text{curl} \omega)^2 \rangle \equiv -2\nu \int k^4 E(k) dk. \quad (3.28)$$

Let us consider the $lim \rightarrow 0$ which is equivalent to the $lim Re \rightarrow \infty$. Substituting the Kolmogorov spectrum (2.20) into (3.25), while assuming that for $k > k_d$ the spectrum $E(k)$ falls off rapidly, e.g., exponentially, yields:

$$\langle \epsilon \rangle \approx -\nu \langle \epsilon \rangle^{2/3} k_d^{4/3} = O(1). \quad (3.29)$$

The energy dissipation rate is Re independent parameter. Hence for the viscosity and k_d cut-off wave number we have respectively:

$$\nu \approx \langle \epsilon \rangle^{1/3} k_d^{-4/3}, \quad (3.30)$$

and:

$$k_d = \nu^{-1} \approx \nu^{-3/4} \langle \epsilon \rangle^{1/4} \rightarrow \infty. \quad (3.31)$$

In reality the energy spectrum for $k \sim k_d$ not the K41 spectrum for sure. But the relations (3.28)-(3.31) can be seen as definitions of $k_d = \nu^{-1}$, whatever is the energy spectrum in this k - space region.

When $\nu \rightarrow 0$ the expression (3.31) is equivalent to (2.16). Let us note once again that it is the same to take the limit of vanishing viscosity ν , or large integral scale L , or large integral scale velocity

since the only important is the value of Re defined by (0.1). In other words the limit $\nu \rightarrow 0$ should be always understood in a more general sense as the limit $Re \rightarrow \infty$.

At the same time the rate of enstrophy generation would be Reynolds number dependent of order $O(Re^{3/2})$. An important associated quantity often measured in experiment is dimensionless skewness parameter for the velocity derivatives that is defined as follows for HIT model:

$$\begin{aligned}
 S &= - \langle (curl\omega)^2 \rangle / \langle (\partial v_x / \partial x_x)^2 \rangle^{3/2} = \\
 &= C_1 \nu \int k^4 E(k) dk / (\int k^2 E(k) dk)^{3/2}, \quad (3.32)
 \end{aligned}$$

where C_1 is Reynolds independent constant. In K41 theory skewness should be a Reynolds number independent quantity. Estimates based on closures usually result in $S = 0.3 - 0.5$. However experiment and DNS consistently show that is weakly growing with the growth of Re .

Let us notice that the balance of energy would be fulfilled if eddy viscosity in Fourier space is introduced:

$$\nu(k)_{eddy} = \langle \epsilon \rangle^{1/3} C k^{z-2}. \quad (3.33)$$

For a certain unique value of z :

$$\begin{aligned}
 z &= 2/3; \\
 \nu_{eddy}^{K41} &\approx \langle \epsilon \rangle^{1/3} k^{-4/3}. \quad (3.34)
 \end{aligned}$$

The meaning of eddy viscosity is that the nonlinear term in the Navier-Stokes equation generates the cascade like constant energy flux in the space of wavenumbers from larger to the smaller ones. The energy flux due to the nonlinear coupling of a given shell $\{k + \Delta k\}$ with all other shells in the space of wavenumbers brings energy *into* this given shell which is effectively balanced in a steady state by the flow *out* of this shell. This steady state process of constant energy flow in k -space can be interpreted as the balancing of two terms: *in* source supplying energy to the shell in k -space and *out* dissipation of energy away from the shell. This latter can be interpreted as a kind of eddy viscosity acting at every scale. This is exactly the scale dependent eddy viscosity introduced before in (2.11) and now in (3.34).

And the value of the eddy viscosity exponent (3.34) is uniquely defined for the energy spectrum exponent $-5/3$.

It is very useful sometimes to model the nonlinear transfer term in the Navier-Stokes equations as a source and the eddy viscosity action (3.34). To see how this can be done let us write down the Navier-Stokes equations in Fourier space. We will need it later for many purposes. Let us do it here using the space Fourier decomposition only and thus leaving explicit dependence of the velocity field on time. Substituting (2.1) into (1.8) and taking into account (1.4) after some transformations we arrive at the widely used classical form of the Navier-Stokes equations:

$$\partial_t \mathbf{v}_i(\mathbf{k}, t) + 1/2 P_{ijs}(\mathbf{k}) \int d\mathbf{q} \mathbf{v}_j(\mathbf{q}, t) \mathbf{v}_s(\mathbf{k}-\mathbf{q}, t) = F_i(\mathbf{k}, t) - \nu k^2 \mathbf{v}_i(\mathbf{k}, t), \quad (3.35)$$

where $F(\mathbf{k}, t)$ is the Fourier transform of the external forcing F_i and the so-called projector operator:

$$P_{ijs}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2) k_s + (\delta_{is} - k_i k_s / k^2) k_j \quad (3.36)$$

It is not difficult to surmise that the projector operator comes essentially from the solution of (1.4) in Fourier space. From the universality hypothesis it follows that the actual nature of the force is not really important if however it stirs the fluid only at large scales. Then we will choose it, although it is not a necessary choice, as a Gaussian source injecting energy into the flow and fully defined by its correlation function:

$$\begin{aligned} \langle F_i(\mathbf{k}, t) F_i(\mathbf{k}', t) \rangle &= \Phi(k < k_0, t) \delta(\mathbf{k} + \mathbf{k}'); \\ \Phi(k > k_0, t) &\rightarrow 0. \end{aligned} \quad (3.37)$$

This sort of forcing can be likened to the stirring "paddle in a pan" that was introduced in the previous Section 2 as one of the ways of creating turbulence. From (3.35) and taking into account that the source is Gaussian it is easy to derive the equation for the energy spectrum $E(k, t)$. Multiplying (3.35) by $v_i(\mathbf{k}', t)/2$, integrating over the solid angle element $k^2 d\Omega^{(k)} = k^2 d\mathbf{k}/k$ and over $d\mathbf{k}'$, ensemble averaging and respecting the flow homogeneity and isotropy we obtain after doing some elementary work:

$$\partial_t E(k, t) = \Phi(k < k_0, t) + T(k) - 2\nu k^2 E(k), \quad (3.38)$$

where:

$$T(k) = -k^2 P_{ijs} \int d\Omega \int d\mathbf{q} \langle \mathbf{v}(\mathbf{q})\mathbf{v}(\mathbf{k}-\mathbf{q})\mathbf{v}(-\mathbf{k}) \rangle, \quad (3.39)$$

Is the energy transfer term due to the nonlinear interactions. Since the nonlinear terms conserve energy it is generally possible to write as follows:

$$T(k) = \text{div}_k j(k)^E = \partial_k j(k)^E, \quad (3.40)$$

where:

$$\begin{aligned} j(k)^E &= \int_0^k dk' T(k') = \\ &= - \int_0^k dk' k'^2 P_{ijs}(k') \int d\Omega^{(k')} \int d\mathbf{q} \langle \mathbf{v}(\mathbf{q})\mathbf{v}(\mathbf{k}'-\mathbf{q})\mathbf{v}(-\mathbf{k}') \rangle \end{aligned} \quad (3.41)$$

is the energy flux due to the nonlinear interaction in k - space. For steady state turbulence we have:

$$\Phi(k < k_0, t) + T(k) - 2\nu k^2 E(k) = 0 \quad (3.42)$$

and in the inertial range $k \gg k_0 \gg k_d$ approximately with the accuracy of terms of order $O(Re^{-1})$:

$$T(k) \approx 0. \quad (3.43)$$

As long as we are considering the inertial range it should be then satisfied:

$$j(k)^E = \langle \epsilon \rangle = \text{const}. \quad (3.44)$$

The energy flux constancy in the inertial range as before is actually a postulate of the existence of the inertial range with Re independent relevant physical quantities.

It is especially elegant and useful when expressed in physical space. Let us introduce the n - th order velocity structure functions as follows:

$$\langle \Delta v(\mathbf{r})_l^n \rangle = \langle [\mathbf{v}(\mathbf{r}+\mathbf{x}) - \mathbf{v}(\mathbf{x})]_l^n \rangle, \quad (3.45)$$

where the subscript l means the longitudinal projection. The longitudinal structures functions are relatively easy to measure in laboratory and for divergence free isotropic and homogeneous vector fields they

provide significant information about the correlation tensors. In particular, the third order structure function $\langle \Delta v(\mathbf{r}_l^3) \rangle$ provides full information on the general two point third order velocity correlation function. It can be shown doing some rather tedious algebra that the Karman-Howarth relation that is in fact the the relation (3.44) but in physical space (e.g., Monin and Yaglom, 1975) is satisfied.

If we define the energy flux in *k-space* as:

$$\langle j(k)^E \rangle = 2 \langle \epsilon^{1/3} C \int_k^{k+\Delta k} k_{-4/3} k^2 E(k) dk \rangle, \quad (3.46)$$

we can see that with logarithmic accuracy it is selfconsistently the same for any shell $\{k, k + \Delta k\}$ in the inertial range. This is a basic premise of K41 theory, which is likely to remain true and survive in the future theory. This flux generated by the nonlinear term at certain $k = k_d$ by continuity matches the viscous dissipation. So it can be understood in both ways, either as production of small scale turbulence by the larger scales or as dissipation, because eventually it matches the viscous dissipation.²⁹

Let us consider what happens with K41 when a flow is not mirror symmetric. In other words when the ensemble averaged helicity is not zero, the spectra $E(k)$ and $H(k)$. Since they are defined by means of symmetric and antisymmetric part of the velocity spectral tensor they should be independent. However there is still a weak restriction imposed by the usual Schwartz inequality, e.g., Moffatt (1978). Indeed, it is obvious from this inequality that:

$$\langle [\mathbf{v}(\mathbf{k})^* \cdot \boldsymbol{\omega}(\mathbf{k})] + [\mathbf{v}(\mathbf{k}) \cdot \boldsymbol{\omega}^*(\mathbf{k})] \rangle \leq 4 \langle [\mathbf{v}(\mathbf{k}) \cdot \mathbf{v}^*(\mathbf{k})] \rangle \langle [(\boldsymbol{\omega}(\mathbf{k}) \cdot \boldsymbol{\omega}^*(\mathbf{k}))] \rangle. \quad (3.47)$$

Then together with the definitions of $E(k)$, $\Lambda(k)$ and $H(k)$ we derive making simple calculations:

$$|H(k)| \leq 2kE(k). \quad (3.48)$$

This seemingly innocuous inequality results in important conclusions. Assume that turbulence is driven by a most helical external force that

²⁹It should be noted that although the mean values are the same the fluctuations of the energy flux should not beareally identical to the fluctuations of the energy viscous dissipation rate. But usually they are assumed to follow the same scaling laws. Nevertheless, the identification of the two is an assumption.

can only be, for instance by a paddle that injects helicity into the flow. If the reasoning of Kolmogorov is assumed than anyway for the high wavenumbers the energy spectrum will be K41 irrespectively of the driving force at large scales (that is small wavenumbers). But then the maximal value of helicity transmitted to the high wavenumbers is limited by the inequality (3.48), so that $|H(k)| \leq 2kE(k) \sim k^{-4/3}$. In fact, even this is not possible because the viscous dissipation defined by (3.26) would be proportional to $Re^{3/4}$. The dissipation then would reduce the helicity spectrum to the steeper falling power law $\pm H(k) \leq E(k) \propto k^{-5/3}$, where \pm is due to the fact that the force injected helicity can be positive or negative. In this case the helicity absolute value dissipation rate would be the same as the energy dissipation rate (3.26). Whatever is the rate of helicity injection into the flow its spectrum will become universal and just the same as the energy spectrum, even though the helicity mean value can be large at large scales. In other words there is no substantial dynamical influence of statistically mean helicity on turbulence dynamics at small scales. Although the reasoning above may seem rudimentary the result is very general. It can be obtained by more elaborate methods that would add nothing to the substance. In practice large enough mean helicity can delay the turbulence cascade and prevent for a period of time the energy flux reaching the high wavenumbers. But when it happens the dynamical influence will become insignificant. This is why helicity by itself is not really an important quantity for the fundamental structure of turbulence as was pointed out before (see Endnote **b** in the end of the text).

Now let us introduce the formalism for a flow projected into a finite cubic lattice with periodic boundary conditions. This is a model closest to HIT model of turbulence in unbounded flow domain and therefore basic for DNS and numerical experiment into turbulence.

4 Kolmogorov Theory of Homogeneous Isotropic Turbulence (K41): Turbulent Flows on a Lattice

Consider the velocity field inside a cube of edglength $1/2\pi$ and its projection onto a finite cubic lattice inside that cube, for instance, a $3D$ grid with N equidistant lattice points in each direction. The

simplest boundary conditions would be the periodic ones:

$$v_i(x, y, z) = v_i(x + 2\pi m, y + 2\pi n, z + 2\pi l), \quad (4.1)$$

where m, n, l are integers. Denote the coordinates of the grid points by $\mathbf{r}^{(\mathbf{n})}$ where \mathbf{n} is a vector whose components n_i are integers from 1 to N and:

$$x_i = n_i/2\pi N. \quad (4.2)$$

The velocity at a point $\mathbf{r}^{(\mathbf{n})}$ can be expanded in a triple Fourier series as follows:

$$\mathbf{v}(\mathbf{r}^{(\mathbf{m})}) = \sum_{\{\mathbf{k}^{(\mathbf{n})}\}} \mathbf{v}(\mathbf{k}^{(\mathbf{n})}) e^{i(\mathbf{k}^{(\mathbf{n})} \cdot \mathbf{r}^{(\mathbf{n})})}, \quad (4.3)$$

where $\mathbf{k}^{(\mathbf{m})}$ is a vector with components $k_i = n_i$ such that:

$$(\mathbf{k}^{(\mathbf{n})} \cdot \mathbf{r}^{(\mathbf{m})}) = n_i m_i / 2\pi N, \quad (4.4)$$

and where:

$$\sum_{\{\mathbf{k}^{(\mathbf{n})}\}} = \sum_{n_i, n_j, n_l}^N. \quad (4.5)$$

The transformation inverse to (4.3) is:

$$\mathbf{v}(\mathbf{k}^{(\mathbf{n})}) = N^{-3} \sum_{\{\mathbf{r}^{(\mathbf{n})}\}} \mathbf{r}(\mathbf{k}^{(\mathbf{n})}) e^{i(\mathbf{k}^{(\mathbf{n})} \cdot \mathbf{r}^{(\mathbf{n})})}, \quad (4.6)$$

where:

$$\sum_{\{\mathbf{r}^{(\mathbf{n})}\}} = \sum_{n_i, n_j, n_l}^N. \quad (4.7)$$

The shell averaged energy spectrum hence is:

$$E^s(k^{(n)}) = 1/2 \sum_{\mathbf{k}^{(n)}/k^{(n)}} (\mathbf{v}(\mathbf{k}^{(n)}) \cdot \mathbf{v}(-\mathbf{k}^{(n)})), \quad (4.8)$$

where the summation is over all the orientations of the unit vector $\mathbf{k}^{(n)}/k^{(n)}$. The summation in (4.8) is called shell averaging and the

energy spectrum itself is shell averaged energy spectrum. This is associated with the energy spectrum $E(k)$. Using the Parseval theorem it follows:

$$E = \sum_{\{k^{(n)}\}} E^s(k^{(n)}) = 1/2N^3 \sum_{\{\mathbf{r}^{(n)}\}} \mathbf{v}(\mathbf{r}^{(n)})^2, \quad (4.9)$$

where the sum is over all possible lengths of $k^{(n)}$. In the same manner we introduce the spectral tensor $F_{ij}^s(k)(k^{(n)})$ which is the analogue of $F_{ij}(k)$ and, in particular, the shell averaged helicity spectrum:

$$H^s(k^{(n)}) = \sum_{\mathbf{k}^{(n)}/k^{(n)}} (\mathbf{v}(\mathbf{k}^{(n)})\omega(-\mathbf{k}^{(n)})), \quad (4.10)$$

which is the analogue of $H(k)$ for the continuous case.

If the number of grid points N^3 is big enough, or in other words the resolution of DNS is high enough the space averaging or the shell averaging in the Fourier space would adequately approximate the ensemble averaging. This is not, however, invariably the case. Since N is finite there will always be a random deviation of the space and shell averages from the genuine ensemble averages. For instance as a result of the statistically independent fluctuations in $E^s(k^{(n)})$ the mean square deviation from the shell and space averaging will be of order $O(N^{-1/2})$. Of course the real dynamics of the Navier-Stokes equation may somewhat alter this estimate but this is not the issue now. What is important to emphasize is that although for most quantities the deviations would be very small they are never small for the quantities for which the ensemble mean is zero. For instance for the helicity and helicity spectrum the space averages never are the same as the genuine ensemble averages, which for mirror symmetric flow are identically zero. Although the real fluctuations of helicity are determined by the dynamics of the Navier-Stokes equations and may not be statistically independent the issue is that quasi-ergodic hypothesis should be applied with caution. In this sense it is more reliable to consider many time realizations of forced BigBox flow.

The time realizations are either generated from many initial conditions or by considering forced turbulence. The forced turbulence means that the force $\mathbf{F} \neq 0$ in Eqs. (1.1). The force can be arbitrary, in particular a useful and popular choice is a random force,

but with one important constraint. It should force only certain large scale harmonics of the flow and not affect directly the small scales so that not to contaminate the naturally developing turbulence cascade. The force should be in other words similar to a paddle that we used as a way to stir the flow in a pan when the K41 model was discussed above. The arbitrariness in the way the forcing is chosen is based on our conviction that the ensuing turbulent flow does not depend in its main features on how turbulence is triggered; to be sure a principle superseding K 41 theory in generality and most probably surviving in the asymptotic limit of $Re \rightarrow \infty$, even if the K41 theory itself is not complete. In DNS the force will result in a flow that when averaged over sufficient number of time realizations and the averaging over space in each time realizations would be an adequate approximation to the ensemble averaged flows with stable steady state features like the energy spectrum. If there are coherent sub-domains in such a flow they should survive this double averaging and show up unambiguously. This all is especially relevant because intermittency and coherent sub-domains are located in very small parts of the fluid volume, which in discretized flow description corresponds to sets with relatively few lattice points $N_F^3/N^3 \ll 1$.

Let us calculate the ensemble mean square deviation of the shell averaged helicity spectrum. The ensemble mean $\langle H(\mathbf{k}) \rangle = \langle H^s, (k) \rangle = 0$, where the superscripts are omitted for simplicity. But the shell averaged $H(k)^s \neq 0$.³⁰ Let us assume that the fluctuations of $H(\mathbf{k})$ are statistically independent. Then evidently we are seeking the following quantity:

$$\sigma_H^2 = \langle [H(k)^{sh}]^2 \rangle = \langle \left[\sum_{\mathbf{k}/k} H(\mathbf{k}) \right]^2 \rangle = \sum_{\mathbf{k}/k} \sum_{\mathbf{k}'/k'} \langle H(\mathbf{k}) H(\mathbf{k}') \rangle . \tag{4.11}$$

Now we use the statistical independence of $H(\mathbf{k})$:

$$\langle H(\mathbf{k}) H(\mathbf{k}') \rangle = \langle H(\mathbf{k})^2 \rangle \delta_{k, k'} . \tag{4.12}$$

We find for the variance (4.11):

$$\sigma_H^2 = \sum_{\mathbf{k}/k} \langle H(\mathbf{k})^2 \rangle . \tag{4.13}$$

³⁰When it is not confusing the superscripts for the wavenumbers will be omitted.

It should be noted that the zero, or in practice weak correlations assumption is equivalent roughly to assuming the Gaussian distribution for the fluctuations. It is more proper to say that a quasi-Gaussian assumption is made, because the strictly Gaussian assumption would lead to zero interaction between the velocity harmonics and subsequently no dynamics at all. Using the Eq. (3.17), which is of course the same for the discrete description we obtain the factorized expression:

$$\sigma_{HG}^2 = \sum_{\mathbf{k}/k} 4k^2 \langle [Re\mathbf{v}(\mathbf{k})^2][Im\mathbf{v}(\mathbf{k})^2] \rangle \langle \sin^2\alpha(\mathbf{k}) \rangle, \quad (4.14)$$

where we have assumed that the angles between $Re\mathbf{v}(\mathbf{k})$ and $Im\mathbf{v}(\mathbf{k})$, actually the phases $\alpha(\mathbf{k})$, are statistically uncoupled from the absolute magnitudes of these vectors, an assumption subsequent to the assumption of statistical independence of $H(\mathbf{k})$ fluctuations. Also it is obvious that:

$$\langle Re\mathbf{v}(\mathbf{k})^2 \rangle = \langle Im\mathbf{v}(\mathbf{k})^2 \rangle = 1/2 \langle E(\mathbf{k}) \rangle. \quad (4.15)$$

Hence we obtain:

$$\sigma_{HG}^2 = 2k^2 \sum_{\mathbf{k}/k} \langle E(\mathbf{k})^2 \rangle, \quad (4.16)$$

where the additional subscript G means Gaussian assumption. Finally assuming that statistically there is rotational isotropy making possible replacing the summation $\sum_{\mathbf{k}/k}$ by $4\pi k^2$, one obtains for the helicity spectrum variance:

$$\sigma_{HG}^2 = 1/2\pi \langle [E(\mathbf{k})^2] \rangle. \quad (4.17)$$

This is a fairly remarkable result in that it is independent of the resolution N . It shows that in any realization of turbulence the shell averaged spectrum fluctuates primarily within the following interval:

$$-E^s(k)/\sqrt{2\pi} \leq H(k)^{sh} \leq +E^s(k)/\sqrt{2\pi}. \quad (4.18)$$

In the next section it will be explained that the assumption of statistical independence of $H(\mathbf{k})$ that is roughly equivalent to the assumption of statistical independence of the phases $\alpha(\mathbf{k}, t)$ is inconsistent

with the dynamics of the Navier-Stokes equations and obstructs the energy cascade to small scales. It should be pointed out that even though K41 theory is not complete but the general concept of the energy transfer to and dissipation at small scales is undoubtedly a correct vision of turbulence. Thus it should be always in our mind for testing the validity of assumptions. If an assumption contradicts this basic tenet than it is definitely wrong. Since the assumption of statistically independent helicity harmonics fluctuations are not compatible with the unimpeded flow of energy from large scales to the small ones than this assumption is wrong. If it is wrong it means that the helicity harmonics $H(\mathbf{k})$ are coherent. This means in fact that $\alpha(\mathbf{k})$ phases are coherent. It is this phase coherence that results immediately in intermittency in physical space. It is a very general property of phase coherence of fields in Fourier space that it transforms and shows as a certain bunching effect for the large amplitudes of this field in physical space, i.e., in general sense intermittency. Thus the result that we must anticipate is that the helicity fluctuations are connected with turbulence intermittency and with no such intermittency the unimpeded energy flow to high wavenumbers is not possible. It is obvious without repeating that all the formulas of the previous sections are correct for the discrete description with integrals and volumes substituted by the sums and number of lattice points. The natural condition for a good resolution of DNS is as follows:

$$N^3 \gg N_{modes}, \quad (4.19)$$

where N_{modes} is defined by (2.22). This condition shows what enormous computational power would be required for faithful DNS of turbulent flows for large values of the Reynolds number. This illustrates the desperate need of geophysical community to have model equations instead of the real Navier-Stokes equations that would eliminate most of the degrees of freedom but at the same time do not "throw a child out of the basin together with water", speaking figuratively. Such models may be possible but in the first place there must be if a not an analytical theory, such as we were used to in other physical disciplines of the past, then at least a good qualitative comprehension of what turbulence really is.

In what follows the discrete and continuous description will be used wherever convenient without further comments and considered

equivalent. Nevertheless, it should be emphasized that there is a principal and not only practical significance in the relations (2.22) and (4.20). What they tell us is that despite the fact that the fluid flow as described by the partial differential Navier-Stokes equations is a space and time continuous problem with generally infinite number of degrees of freedom turbulence is always a problem with finite number of degrees of freedom proportional to the finite phase volume k_d^3 that only asymptotically tends to infinity together with the unbounded growth of the Reynolds number. In this there is clear similarity with other nonlinear systems with finite degrees of freedom and complicated chaotic dynamics and herewith lies hope that may be much smaller phase space and much smaller number of degrees of freedom would be sufficient to describe turbulence faithfully.

5 Kolmogorov Theory of Homogeneous Isotropic Turbulence (K41): What is Correct and What is Wrong

In fact the discussion was started in the previous Section 4. But let us go back to the historical roots that should help better understanding of the facts that have been gradually established since K41 theory release. It was soon noticed that the reasoning leading to the K41 spectrum is not unique. Indeed, from dimensional considerations one can use $\langle \epsilon \rangle^{2/3}$ for the derivation of the spectral function (2.20). But equally it can be local $\epsilon(\mathbf{r}, t)$ and not the averaged $\langle \epsilon \rangle$ used from dimensional considerations. But $\epsilon(\mathbf{r}, t)$ is itself a fluctuating field. What it means is that dimensionally we could use for the spectrum definition, say $\langle \epsilon^n \rangle^{2/3n}$, where n is arbitrary.³¹ There are no physical or mathematical reasons to choose $n = 1$. If ϵ was similar to a normal statistical type quantity fluctuating as a Gaussian variable this would be not a significant change affecting only a constant in front of the spectrum. However it was early noticed by Batchelor and Townsend (1949) that this is not the case. In fact is quite a wildly fluctuating quantity that reminds nothing of the usual field fluctuations in near to equilibrium statistical systems. In **Fig. 7** borrowed from Kit, et al., (1987) one can see typical experimental

³¹This reasoning is usually attributed to L. Landau and can be found in B. Mandelbrot (1983).

time traces of various quantities measured at a location in turbulent flow past a grid—a velocity component denoted as $u_i(t)$, a vorticity component ω_i , a part of the helicity density $u_i(t)\omega_i(t)$, energy density dissipation $\epsilon(t)$, enstrophy $\omega^2(t)$ and other more nonlinear quantities relevant for turbulent dynamics. Inspecting the time traces immediately shows that for instance the velocity component varies in time in what appears a relatively smooth Gaussian like manner. But the energy dissipation and enstrophy time traces are totally different. They consist of quiescent long signals with quite rare distinct large amplitude peaks at approximately the same locations on the time axis for the two fields. Even more distinct peaks are evident for other quantities depicted in the **Fig. 4** that are cubic nonlinearities with respect to the velocity field derivatives.

It is clear that dependent on n the averages $\langle \epsilon^n \rangle^{2/3n}$ would be completely different and in consequence the energy spectrum may be different. Since there is no reason to choose one n or another for the derivation of the energy spectrum the whole sequence of considerations that led to K41 theory crumbles.

The velocity fluctuations in turbulent flows are primarily determined by large scale or low wavenumbers velocity harmonics as is seen from Eq. (2.24). In contrast the fluctuations of the velocity derivatives in general and the enstrophy, its rate of growth and energy dissipation rate $\langle \epsilon(t) \rangle$ are all strongly skewed to small scales or high wavenumbers values, see for instance Eqs. (2.23), (3.28) and (3.29). Note that the last property would remain the same if the energy spectrum is not K41 but another power law close enough to K41. The moments of the energy dissipation rate $\langle \epsilon^n \rangle^{2/3n}$ are definitely anomalous by comparison with what is regarded as normal in statistical analysis that is the moments as they should have been if their fluctuations followed the normal Gaussian (bell curve) distribution. In fact for big enough they are defined primarily by the rare peaks of intensity in the time traces and this dominance increases with the growth of n . But at the same time this anomalous growth can be only due to the high wave numbers harmonics of $\epsilon(t)$. Together it means that the high wave numbers harmonics of the energy dissipation $\epsilon(t)$ are located in the time trace amplitude peaks or equivalently in narrow bands of physical space flow volume.³² The same is true

³²The identification of spatial patches and temporal peaks requires adoption of

for all other velocity derivatives related quantities such as enstrophy, enstrophy generation, etc., as can be clearly seen in **Fig. 7** below.

So far it was the usual description of intermittency familiar to everyone studying turbulence and chaos in dynamical systems. But the reality is that these intermittent patches of activity are built of coherent helical cells with near to maximal helicity and besides forming obvious clusters of organized vortical activity. The coherent objects that I named **BCC** above. The recognition of **BCC** indicates a totally new reality of understanding intermittency in turbulence.

Let us try to develop this understanding step by step. To start with after years of experimental study it is rather safe to believe that itself fluctuates in accordance with a scaling power law as follows, e.g., Monin and Yaglom (1975):

$$\langle (\epsilon(0) - \langle \epsilon \rangle)^2 \rangle = \langle \epsilon \rangle (L/r)^{\mu_\epsilon}, \quad (5.1)$$

where L is the integral scale and the experimental value of parameter μ_ϵ are not really precise, but probably in the interval $0.3 < \mu_\epsilon < 0.5$. In particular for $r = l_d$ one obtains from (5.1) the mean square deviation of $\epsilon(t)$ that is actually measured in experiments:

$$\langle (\epsilon(0) - \langle \epsilon \rangle)^2 \rangle = \langle \epsilon \rangle (L/l_d)^{\mu_\epsilon} = \langle \epsilon \rangle Re^{3\mu_\epsilon/4}, \quad (5.2)$$

Other quantities often measured in the laboratories are the so-called structure functions of longitudinal velocity projections.³³ These are defined as follows:

$$\langle \Delta \mathbf{v}(\mathbf{r})_l^n \rangle = \langle [\mathbf{v}(\mathbf{r}) - \mathbf{v}(0)]_l^n \rangle \propto \langle \epsilon \rangle^{n/3} (r/L)^{n/3 - \mu_\nu(n)}, \quad (5.3)$$

where the subscript l means longitudinal projection. The meaning of the relation (5.3) is like this. If K41 is correct in its original form then $\mu_\epsilon = 0$. This is because the original K41 theory ignores the anomalously large fluctuations of small scale velocity variance and the related quantities, such as space and time derivatives of $\mathbf{v}(\mathbf{r}, t)$ and their powers. In other words the original K41 ignores intermittency. This would mean that $\langle \mathbf{v}(\mathbf{r})_l^n \rangle = \langle \epsilon \rangle^{n/3} r^{n/3}$. But experiment

a rudimentary ergodicity hypothesis.

³³Only the longitudinal projections of the velocity variations parallel to the hot wire are usually measured in experiment. The only exceptions that I am aware are the experimental works cited above (see also Endnote *g*).

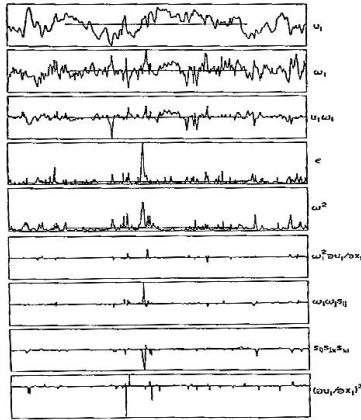


Figure 7: Laboratory measurements of time series of various turbulent quantities in turbulence past the grid from Kit, et al., (1987). From top to bottom they are one velocity component u_1 , one vorticity component ω_1 , $u_1\omega_1$, energy dissipation, rate, enstrophy ω^2 , part of the vorticity stretching term, $s_{ij} = 2e_{ij}$, see Eq. (1.18), etc. We chose these particular measurements because they were done in a unique set of experiments in turbulent electrolyte. This allowed direct measurements of the quantities in a manner different to the usual hot wire anemometer measurements in turbulence that necessitate further application of various assumptions in order to arrive at the quantities of physical interest. It can be clearly seen that the fluctuations of velocity are mild, near to Gaussian, vorticity is much more intermittent with distinct peaks separated by long stretches of mild fluctuations, which is seen even better from observing fluctuations of enstrophy. The higher is the power of velocity derivatives the more obvious are the rare peaks of intensity separated by quiescent periods and this is the characteristics of intermittency. The higher are the orders of moments the more dominant will be the contribution of the rare peaks of intensity of the respective quantities.

firmly shows that although $\mu_\nu(n) \approx 0$ for $n = 2$ with experimental accuracy, which is equivalent to the Kolmogorov law (2.14), $\mu_n u(n) \neq$

0 and is nonlinear at least for $2 < n \leq 6$.³⁴ Intermittency thus conforms with generalized scaling.

This latter is a relaxed version of the K41 theory that stated that the properties of turbulence in the inertial range should be independent of both the Reynolds number, in the limit of $Re \rightarrow \infty$ and the integral scale L . If the first and more profound scaling assumption is still in place but the second is relaxed there is freedom now to use the L parameter for the scaling power laws of the type of (5.1) and (5.3). In other words the velocity structure functions become more and more singular for small separation scales $r \rightarrow 0$, i.e., or in general sense for high wavenumbers. This is in accord with the vision that the intense part of turbulence connected with turbulent field velocity variations and derivatives is concentrated in progressively smaller and smaller sub-domains in the fluid volume.³⁵ Thus it seems that although the original K41 theory is not really correct nevertheless the generalized scaling concept and universality of turbulence, at least HIT, are confirmed experimentally with reasonable experimental evidence.

The above led to great efforts of building phenomenological models that would put together K41 theory and intermittency. The generalized scaling theories inevitably lead to fractal and multifractal models of turbulence (Mandelbrot 1977, 1982, 1983). Let us consider for methodological purposes the simplest of them due to Novikov and Stewart (1964), which is known in literature as fractally homogeneous turbulence-FHT. While doing this we will introduce some important definitions that will be used for the exposition of a dynamical theory later in this paper.

Consider a box of size L that is split into sub-boxes of size l_1 . If we measure the energy flux $\epsilon(\mathbf{r}, t)$, or better to say observe it with low resolution vision, for instance our sight is clouded by tears and we see no fine details we may conclude that the dissipation is

³⁴Generally n is not integer. The value of $n = 6$ is approximate and seems not universal. The behaviour of $\mu_\nu(n)$ for high n may be linear but this is likely due to the fact that high order moments are spurious and do not have much physical meaning if at all as discussed below.

³⁵Note that μ in (5.1) generally does not coincide with any of $\mu_\nu(n)$ from (5.3), unless additional assumptions are made. The intermittency exponents for the correlation functions in Fourier space are difficult to relate to those in physical space because the corresponding physical space Fourier transforms are not local.

homogeneously distributed everywhere in the whole box of the size L . More precisely we would see the averaged over the cube and time value of the flux $V^{-1}T^{-1} \int \epsilon(\mathbf{r}, t) dV dt$, where V is the full size cube volume and $T = T(V)$ is the corresponding time scale. But when we improve our resolution, wipe mist from our eyes, we would see that in fact the dissipation occurs, may be also homogeneously, but only in $n < m$ of the sub-boxes of size l_1 . In other words we would observe now the flux averaged over only a part of the space/time in a fraction $n/m = (l/L)^\mu < 1$ sub-boxes. If we magnify our resolution further, put on the glasses, we would see that in reality the dissipation occurs in even a smaller fraction of size l_2 such that the ratio $\lambda = l_2/l_1 = l_1/L = const$. The fractally homogeneous set is built by self-similar iterations of this process. Taking account of the constancy of the total energy flux through the space sub-boxes with increasingly finer resolution the following relation follows:

$$\epsilon(L) = \epsilon(l_1)n/m = \epsilon(l_2)(n/m)^2 = \dots = \epsilon(l_i)(n/m)^{l_i}, \quad (5.4)$$

or:

$$\epsilon_i = \epsilon_{i-1}\lambda^{-\mu} = \epsilon_0(l_1/L)^{-\mu}. \quad (5.5)$$

The effective volume through which the energy flux is passing (and eventually becomes equal to the viscous rate of energy dissipation at the scales of order l_d) is determined in a similar way with the result:

$$V_i = L^3(n/m)^i = L^3(l_1/L)^\mu(l_2/l_1)^\mu \dots (l_i/l_{i-1})^\mu = L^3(l_1/L)^\mu. \quad (5.6)$$

Now it is easy to calculate the moments of the energy flux:

$$\langle (\epsilon(0) - \langle \epsilon \rangle) (\epsilon(r) - \langle \epsilon \rangle)^n \rangle = \langle \epsilon \rangle^n (L/l_1)^{(n-1)\mu}; r \approx 1. \quad (5.7)$$

In particular for $n = 2$ setting $l_1 = l_d$ we obtain the scaling relation (5.2). We built a simplest possible fractal model with the "volume" tending to zero with $l_1/L \rightarrow l_d/L \rightarrow 0$ as $Re^{-3\mu/4} \rightarrow 0$, while the volume of the active sub-domain tends to zero as $L^3 Re^{3\mu/4}$, the area of the surface bounding this sub-domain will tend to infinity as $L^2 Re^{3\mu/4}$. The actual dimension of the sub-domain is a fractal $D_F = D - \mu$ with $D = 3$ in this case.

Let us consider the above a bit differently in a way convenient for the further exposition in the sections below. Let us associate a

material volume with an active sub-domain embedded in $3D$ flow domain. Let us make an infinitesimal scale transformation $\mathbf{r}' = e^{-l}\mathbf{r}$, where we choose $l \rightarrow 0$ and positive. By definition of a fractal set of dimension $D_F = D - \mu$ embedded in $3D$ space, the material volume of a small sub-domain under this scale transformation becomes $\Delta^{D_F} \mathbf{r} = e^{-(D-\mu)l} \Delta^D \mathbf{r}$. If the fractal is isotropic it means that the effective contraction under the scale transformation is the same in x, y, z directions, i.e., $\mathbf{r} \rightarrow e^{-\mu l/3} \mathbf{r}' = \mathbf{r} e^{-(D-\mu)l/3}$. Let us iterate the rescaling $m \rightarrow \infty$ times. Since m is arbitrary let us choose it in such a way that $\lim_{m \rightarrow \infty, l \rightarrow 0} e^{-ml} = \ln(r/l_d)$. Then asymptotically the iteration of infinitesimal scaling transformation results in the following:

$$\Delta^D \mathbf{r} \rightarrow \Delta^{D_F} \mathbf{r} = \Delta^{D-\mu} \mathbf{r} l_d^\mu, \tag{5.8}$$

while

$$\mathbf{r} \rightarrow \mathbf{r}^{(D-\mu)/3} l_d^{\mu/3}.$$

But it can be that the fractal is strongly anisotropic. In the extreme case the scaling will look like this:^m

$$\begin{aligned} \Delta^D \mathbf{r} &\rightarrow \Delta^{D_F} \mathbf{r} = \Delta \mathbf{r}^{(D-\mu)} l_d^\mu, \\ (x, y) &\rightarrow (x, y)', z \rightarrow z^{(D-\mu)} l_d^\mu. \end{aligned} \tag{5.9}$$

Let us associate a physical field with the material volume, for instance the energy flux. Since the latter is conserved under the scaling transformation (5.8) we obviously arrive at (5.7). But what happens with the velocity field $\mathbf{v}(\mathbf{r}, t)$ and other fields that are not conserved under the scale transformation (5.8) or (5.9)? In the FHT it is easy to calculate. One of the few exact consequences of the Navier-Stokes equations is a fairly remarkable relation of Karman-Howarth-Dryden for the velocity structure function of order $n = 3$ (see also (2.10)). If statistical homogeneity and isotropy are assumed then it can be derived for the separation distances r from the inertial range and in the limit $Re \rightarrow \infty$ (e.g., Monin and Yaglom, 1975):

$$\langle \Delta v^3 \rangle = 4/5 \langle \epsilon \rangle r. \tag{5.10}$$

Since the cascade remains homogeneous at each cascade subdivision in (5.4) it is natural to suppose that the same relation is preserved at each scale. Then it is easy to derive that the exponents in (5.3)

are linear and all determined by one exponent μ defining the square fluctuations of $\epsilon(\mathbf{r},)$:

$$\mu_\nu(n) = \mu(n - 3)/3, \quad (5.11)$$

implying corrections to the second order structure function at $n = 2$ and subsequently the K41 spectrum. FHT is the simplest fractal model of turbulence that can only be by the criteria of our times. More complicated phenomenological so-called lognormal model was built by Kolmogorov and Obukhov. In the lognormal model (e.g., Monin and Yaglom, 1975) the intermittency parameter $\mu(n)$ is nonlinear. Realization that the lognormal model of turbulence is a particular case of more general class of multifractal models of turbulence came much later (e.g., Parisi, Frisch, 1985). For quantitative and deep analysis one should go to Mandelbrot monograph and the works of many authors cited therein. For the purposes of this paper the most relevant analysis of multifractals, their conceptual foundations and relevance for turbulence, geophysics and meteorology have been developed in the seminal works of Lovejoy, Schertzer and their co-authors whom I cite often in this paper (e.g., Lovejoy, 1982; Lovejoy and Schertzer, 1985; Lovejoy, et., al., 2007; Lovejoy, et., al., 2008; Lilley, et., al., 2008; Schertzer and Lovejoy, 1983, Schertzer and Lovejoy, 1985a; Schertzer and Lovejoy, 1985b, etc.). Most of their works can be conveniently found on the website of GANG-Group for the Analysis of Nonlinear Variability in Geophysics.

It seems beyond reasonable doubt that based on the analysis of extensive geophysical data, airborne, satellite, radar and recently lidar that all the atmospheric turbulent fields of importance, turbulent wind in horizontal and vertical, rainfall, admixtures distribution in atmosphere, etc., have universal scaling multifractal organization. The more extreme are the deviations of these fields from the mean the smaller is the dimensional support for these extremals. This scaling organization extends from the smallest scale of millimeters to apparently the largest planetary scales spanning ten orders of magnitude, a truly remarkable conclusion contradicting all the previous geophysical concepts. At the same time it seems that there is no end to the extreme deviations of turbulent fields from their mean values. Most probably the Pdf for the extreme fluctuations is a power law; to be sure asymptotically in the limit of $Re \rightarrow \infty$. What it means is that

that the high order statistical moments of the fluctuating turbulent fields are divergent and actually do not have physical meaning in this limit. Since experiment is always done for finite values of the measurements would show finite values for all orders of statistical moments. But these finite values are spurious and do not have sense beyond the fact that the moments are dominated by a few extreme deviations from the mean. This is good news for practitioners of turbulence. Because if multifractal structure with all meaningful moments of the velocity field and its derivatives is correct then generally turbulence must be characterized by an infinity of scaling exponents for different physical fields and their different order moments. This would render the whole concept quite useless and would forever make turbulence intractable to any reasonable scientific analysis. In fact the realization that only a finite number of moments of physical fields in turbulent flows have meaning makes it possible trying to determine their dynamical significance and building a dynamical theory.

In the framework of helical structures concept the statistical description of turbulence is limited and it seems clear that high order statistics should be meaningless. Indeed, what is the meaning of the statistical moments if the main contribution to these moments comes from **BCC**, an immensely coherent and asymptotically fractal object? So far the singularities of high order moments of turbulent fields have not been seen in laboratory experiments, e.g., Sreenivasan and Antonia (1991), but this does not at all eliminate the concept's validity since the Reynolds numbers of laboratory measurements, the precise ones, are incomparably lower than in geophysical flows. On the other hand it is increasingly indicative from DNS that reaching the real scaling is a slow asymptotic process requiring high values of *Re*. Also, the local in space nature of laboratory measurements are not conducive for capturing the extreme fluctuations that may occupy a very small physical sub-domain in space/time.ⁿ

Let us come back to the role of helicity. The HIT and statistical mirror symmetry are assumed below, meaning that $\langle h \rangle = \langle H(k) \rangle = 0$. Let us consider the relation for the variance (4.11). By definition it follows:

$$\partial/\partial t \sum_k \sigma_H^2 \equiv \partial/\partial t \left[\sum_k H(\mathbf{k}) \right] \left[\sum_{k'} H(\mathbf{k}') \right] > . \quad (5.12)$$

But each of the sums in Eq. (5.12) is just the total helicity, not to be confused with the ensemble averaged helicity, which is assumed to be zero, and therefore conserved by the nonlinear term in the Navier-Stokes equations. Then of course $I = \sum_k I(k) = \sum_k \sigma_H^2$ is also conserved by the nonlinear term, e.g., Levich (1987). Therefore we obtain for the I - *invariant*, setting here $\mathbf{F} = \mathbf{0}$ in (1.1):

$$\partial I \partial t = -2\nu \sum k k^2 I(k). \quad (5.13)$$

Assume now that the helicity fluctuations are not or weakly correlated. Then for $I(k) = \sigma_H^2$ we can use the factorized expression (4.17) with the result:

$$\begin{aligned} \partial I \partial t &= -2\nu \sum k k^2 I(k) \sim -\nu \sum k k^2 E(k)^2 \sim \\ &\sim -\nu \sum k k^{-4/3} = O(-k_d^{-4/3} k_d^{-1/3}) = O(k_d^{-5/3}), \end{aligned} \quad (5.14)$$

where we used the K41 $E(k) \propto k^{-5/3}$ and (3.30) for ν . When $\nu \rightarrow 0$ or $k_d \rightarrow \infty$, which is the same as $Re \rightarrow \infty$, the rate of dissipation of I - *invariant* tends to zero. On the other hand the I - *invariant* in the Gaussian approximation is predominantly determined by the large scale helicity fluctuations. Indeed, using (4.17):

$$I = \sum_k I(k) \equiv 1/N^3 \sum_r \langle h(\mathbf{r})h(0) \rangle = \sum_k I(k) \propto L^{7/3}. \quad (5.15)$$

In other words the large scale helicity fluctuations do not dissipate. But it is at these scales that most of energy resides. Then clearly the cascade of energy to small scales is put on brakes and can be shown stopped.³⁶

What was demonstrated above is that the assumption of uncorrelated helicity fluctuations at high wavenumbers, an innocuous assumption it would seem at the first glance, results in contradiction with the reality true for all turbulent flows-the free energy cascade

³⁶Or it is necessary to have the inverse cascade of I to large scales that will pull away most away most energy from the cascade to small scales. For reference I would like to mention that similar reasoning can be applied to MHD turbulence, but with substitution of helicity by cross helicity, $A = \int \mathbf{v} \cdot \mathbf{B} dV$, (Levich and Shtilman, 1982).

to small scales and subsequent viscous dissipation. The conclusion is that the assumption is wrong and should be expunged. Note that the exact form of the energy power law spectrum is not important for this conclusion and small scaling corrections to K41, in the unlikely case of their existence, would not change the conclusion.

The assumption that should be made instead is that on the contrary the small scale $H(\mathbf{k})$ harmonics are strongly correlated in such a way that the rate of viscous dissipation of I - *invariant* is the same as the rate of energy dissipation. For this to happen it is necessary to assume as follows:

$$I(k) \propto I(k)_G (L/k)^{\mu_H} \propto E(k), \quad (5.16)$$

where if the K41 spectrum is assumed the intermittency exponent $\mu_H = 5/3$.³⁷ This choice of the intermittency exponent would make the rate helicity fluctuations rate of dissipation (5.14) and the energy rate of dissipation (3.25) of the same order $O(1)$, i.e., finite and the Reynolds number independent. But this intermittency exponent $\mu_H = 5/3$ is large and indicates very strong $\alpha(\mathbf{k})$ phase coherence (see the definition (3.17) and the considerations leading to (4.14)). The phase coherence means bunching of the corresponding field in physical space or intermittency. And what is the corresponding field in physical space? The Fourier inverse of $H(\mathbf{k})$ can be written as follows:

$$\phi(\mathbf{r}) = N^{-3} \sum_{\mathbf{r}'} \mathbf{v}(\mathbf{r}') \cdot \omega(\mathbf{r}-\mathbf{r}') \equiv N^{-3} \sum_{\Delta\mathbf{r}} \mathbf{v}(\mathbf{r} + \Delta\mathbf{r}/2) \cdot \omega(\mathbf{r} - \Delta\mathbf{r}/2), \quad (5.17)$$

where a convenient redefinition of variables was made: $\mathbf{r}' \rightarrow \mathbf{r} + \Delta\mathbf{r}/2$. The high wavenumbers coherence of $H(\mathbf{k})$ fluctuations is necessarily a bunching, or wave packets of $\phi()$ in physical space with high amplitudes in small sub-domains containing high wavenumber harmonics of $H(\mathbf{k})$. As the wavenumber $k \rightarrow \infty$ the sizes of the sub-domains

³⁷Note however that since only near to dissipation and dissipation range scales are important for the integral rate of dissipation the assumption that in this range the energy spectrum is K41 is not justified. Thus the value of intermittency exponent is illustrative rather than a prediction. In fact, there are grounds to believe that in near to dissipation range the energy spectrum is flatter than K41, as was asserted in Levich (1987) and now seems to be compatible with DNS of Mininni et al., (2008b)

of $\phi(\mathbf{r})$ will tend to zero as $\Delta r \sim k^{-1}$. But it is obvious from the definition of $\phi(\mathbf{r})$ that the high amplitudes means strongly correlated \mathbf{v} and ω inside the sub-domains. The phase correlations are all important. Indeed the variation of velocity occurs on a time scale commensurate with the large scale motion, since velocity is dominated by the large scale harmonics. But vorticity is dominated by the small scale harmonics of order $k \leq k_d$. Thus they vary on a small time scale generally unless constrained by the coherence as they are in strongly helical flows. The correlations in both Fourier and physical space are totally determined by the correlations of phases, i.e., in this case the angles $\alpha(\mathbf{r}, t)$ between velocity and vorticity and the angles $\alpha(\mathbf{k}, t)$ between $Re\mathbf{v}(\mathbf{k}, t)$ and $Im\mathbf{v}(\mathbf{k}, t)$.

It is convenient to interpret $\phi(\mathbf{r})$ as a topological "charge" in a sub-domain averaged over its volume $\sim \Delta r^3$. The maximal possible charge for given amplitudes $|\mathbf{v}|$ and $|\omega|$ would be given if there is a Beltrami flow inside this sub-domain. And the bunching of Beltrami like topological charges is in relation with anomalously large values of the correlation function:

$$\gamma = \langle \phi(\mathbf{r})\phi(\mathbf{r} + \Delta\mathbf{r}) \rangle, \quad (5.18)$$

by comparison with what would be their quasi-Gaussian values, for small separation lengths Δ from the inertial range. If scaling is assumed this would mean that statistically for many realizations the sizes of the sub-domains can be any from the inertial range.³⁸ Since the mean helicity is zero the volume averaged helicity should be at least small for any volume much bigger than the size of the sub-domains. But since the sub-domains are of any size from the inertial range it means that each of the Beltrami sub-domains should have nearby (statistically) an equivalent anti-Beltrami sub-domain with the opposite helicity sign. This is the reason why these topological charges should cluster together rather than be distributed unconnectedly. Alternatively they may have a short life time, virtual in a sense, so that not to result in violation of mean helicity conservation law. It

³⁸It is useful to get used to the asymptotic manner of perceiving the inertial range. It is always correct to think that the wavenumbers tend to infinity in this range because of the inertial range definition (2.21) with L finite and $Re \rightarrow \infty$. This asymptotic manner of thinking is of course general for all the phenomena described by scaling laws

should be noticed that on a qualitative level we arrived at the **BCC** concept.

It may be rather strange at the first glance to the ones used to the K41 theory because the above reasoning was all based in order to salvage the principles of universality, scaling and energy flow to high wavenumbers which is usually associated with K41. But the conclusion is that turbulence coherence is not in contradiction with these general principles but on the contrary the necessary condition for them to be true. Neither the coherence is in conflict with K41 energy spectrum, but on the contrary the K41 energy spectrum is totally impossible in the incoherent turbulent scenario. This is a point not easy to accept by some of us since our education was that the spectrum is derived from the incoherent vision of turbulence and many years of work by many were dedicated to find ways to rescue this spectrum despite the obvious experimental intermittency and coherence. And yet the **BCC** scenario is not only compatible with the K41 energy spectrum but as will be shown later is most probably the mechanism of generation of this energy spectrum rather than the purely homogeneous cascade phenomenology of K41 theory; that is to say that one should distinguish between the K41 spectrum, which seems correct and the particular mechanism of its formation from the K41 theory, which is highly unlikely to be true. Nevertheless, the principles of scaling and universality are much more general than any particular dynamical mechanism and it is rather they select the required available dynamical mechanism compatible with these principles. In this case this mechanism is the formation of **BCC**.³⁹

³⁹The suggestion that K41 spectrum is formed by vortex sheets in turbulent flows, the remains of surfaces of tangential discontinuities typical for inviscid description, rather than the result of homogeneous energy cascade was made a long time ago by Townsend (1949). But then when the spectrum is restored it turns out that the spectrum would be $\sim k^{-2}$, which is definitely wrong experimentally. However the development of this concept popular among Cambridge school of turbulence was that the sheets are unstable as a result of Kelvin - Helmholtz instability and fold multiplicatively, the phenomenon called curdling after Mandelbrot, and they may create in principle any power law energy spectrum (see: Moffat, 1983). In this general context **BCC** can be seen as an extension of the vision of turbulence by Townsend and Cambridge school. But the intrinsic coherence and universality of **BCC** are new concepts. By the way, in $1D$ analogue of the Navier - Stokes equations called the Burgers equation with a stochastic forcing in the r.h.s., the shock wave like singularities indeed form the spectrum $\sim k^{-2}$. But in $1/D$ case there is no curdling and there is no intrinsic dynamical

Practically it is nearly impossible to observe experimentally in laboratories the helical fluctuations with conventional measurements. Physical experiment in turbulence, it is reminded, is done by hot wire measurements, as a rule, and fundamentally local, at best at several space points. Nevertheless, visualization of CS in many natural flows and especially in geophysical flows leaves no doubt of their ubiquitous presence everywhere in turbulence, as was suggested in Tsinober and Levich (1983) and Levich and Tzvetkov (1984, 1985).⁴⁰ Unambiguously however it can be done by visualization in DNS with high enough resolution as was done by Mininni, et al. (2008a and 2008b), or considering the $Pdf(\cos\theta)$ between \mathbf{v} and $\boldsymbol{\omega}$ as was described previously. But the last one is not easy to interpret correctly as I spent time explaining before. The way to see the effects indirectly otherwise is in Fourier space by calculating $I(k)$ in DNS and comparing with their values that would be if it was assumed that the velocity harmonics fluctuate in the quasi-Gaussian manner. This was done in Levich and Shtilman (1988) and Levich, et al., (1991) and shown in **Fig. 8** below. The model that was considered was the usual forced BigBox turbulence with a randomly fluctuating Gaussian force \mathbf{F} and the Fourier spectrum fast decaying to zero away from the lowest values of the wavenumbers. This latter is always necessary so that the energy injection is only at the low values of wavenumbers $|\mathbf{k}|$ or the largest scales. This allows the uncontaminated energy cascade to lower scales and the ensuing "steady state" turbulence. Importantly the mean helicity injected by this force when averaged over a span of time was zero. Even though the DNS of those years were for relatively low Reynolds number flows the plot in **Fig. 8** is quite clear: for the low values of wavenumbers dominated by the external forcing the $I(k)$ spectrum is almost the same as its quasi-Gaussian value would have been, but for the high wavenumbers where the flow properties can be expected, if one is based on the principles of universality, to attain independence of the forcing, the $I(k)$ spectrum shows very large amplitudes by comparison with their Gaussian values, as was

chaos. The solution for the spectrum is obtained analytically.

⁴⁰While staying humble in front of the complexity and beauty of turbulence phenomenon it nevertheless should be pointed out that the pervasive helical manifestations become much clearer to an observer if he/she is consciously expecting them. If not one may be watching the same CS and the same clouds in the sky without perceiving or may be not attaching importance to their helical build up.

predicted by the theoretical analysis. The physical space correlations were not calculated because the physical space correlation functions require much finer space resolution due to intermittency: large amplitudes are confined to relatively small domains and projected onto a small number of grid points on a lattice in DNS.

Let us consider now the case of decaying turbulence. In this case turbulence starts from some initial flow as was explained before and this initial flow is allowed to evolve. The energy is very fast transferred to the high wavenumbers and turbulence ensues. Usually in DNS it happens after one large eddy turnover time $\sim \epsilon^{1/3} L^{2/3}$ during which the integral scale eddy with the velocity $\epsilon^{1/3} L^{1/3}$ passes its own length L , or faster (e.g., (2.8)). Usually the initial flow is chosen as a random Gaussian with velocity harmonics excited principally at some large scales or small wavenumbers. The initial state may be chosen as zero helicity at each point for convenience, i.e., $\mathbf{v}(t=0) \cdot \boldsymbol{\omega}(t=0) \equiv 0$. Or it may be any other initial condition as far as the initial helicity is concerned, e.g., $H(\mathbf{k}) = 0$ for all values of \mathbf{k} . The ensuing Navier-Stokes dynamics however is almost entirely independent from the helicity properties of the initial flow. Consider **Fig. 9** in which the helicity spectrum $H^s(k)$ from (4.10) is shown for the developed stage of decaying turbulence.

The spectrum is compared with the analogous shell averaged energy spectrum (4.8). We observe two things. The first one is quite trivial that despite the fact the initial helicity was zero the shell averaged $H^s(k)$ is not. But it was explained above, see the inequality (4.18), that viscosity can and generally generates helicity fluctuations. If such fluctuations were random than they would have had amplitudes defined by the inequality (4.18). But they are not random. And this is the second observation that is not trivial at all. How do we see that the fluctuations are not random? Because in the region of high wavenumbers the $H(\mathbf{k})$ harmonics are obviously such that the shell averaged over all directions of \mathbf{k} sum up having the same sign of $H^s(k)$ for almost all $|\mathbf{k}| = k$ and furthermore for all practical purposes:

$$\pm H^s(k) \approx E^s(k). \quad (5.19)$$

This is instead of randomly fluctuating between positive and negative values for each particular \mathbf{k} in the interval $|H^s(k)| \leq 1/\sqrt{2\pi}E(k)$, as would have been the case if the helicity harmonics were not finely

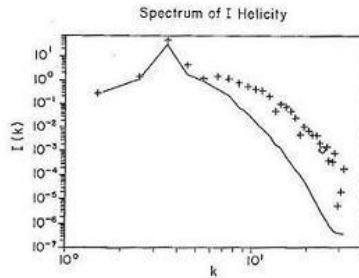


Figure 8: Shows the forced steady state turbulence DNS spectrum of helicity fluctuations defined by Eq. (5.15). The solid line corresponds to the Gaussian helicity fluctuations with no phase coherence defined by Eq. (4.16) and the crosses correspond to the real dynamics from the Navier-Stokes equations. The intermittency, in this case the deviation from the Gaussian spectrum in high wavenumber range is enormous.

correlated. We observe the same sort of phase coherence as the one that resulted in anomalous helicity spectrum variance $I(k)$. But now the coherence is observed on the level of one turbulent realization. It should be noted that we consider only representative realizations with the number of grid points N big enough, ($128 \times 128 \times 128$) in this particular DNS. It means that the deviations of the energy spectrum from the mean for instance are very small of order $O(N^{-1/2})$. In this sense one can make a conclusion that what is qualitatively correct for one representative realization should be true for the ensemble of realizations as well. And the fact that in each (representative) realization the helicity related phases $\alpha(\mathbf{k}, t)$ defined in relation (3.17) are coherent in precisely such a way that makes it possible to satisfy (5.16) for the ensemble of realizations.

Such conclusion was effectively made before in Section 3 but with traditional and wrong emphasis. The second conclusion is that it is the intrinsic *helicity fluctuations* that are all important and not the ensemble averaged helicity which can be zero or small. The fluctuations are the essence of intermittency and eventually **BCC** of which

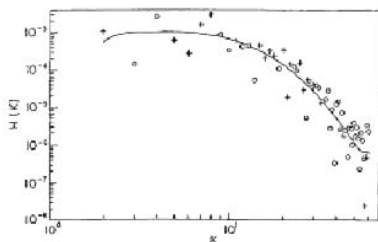


Figure 9: Helicity spectrum in DNS of decaying Big Box turbulence with crosses and circles corresponding to negative and positive values of $H^s(k)$. The circles cluster and clearly prevail over the rare crosses in the high wavenumbers range. This is imperative to have helicity dissipated by viscous term with the same rate as energy is dissipated. This is clear phase coherence. The total helicity is determined by the low wavenumbers. But it is small and not really relevant for the dynamics. It should be reminded that the helicity itself and helicity fluctuations are spontaneous and created by the intrinsic dynamics by viscous force in the Navier-Stokes equations. There is no helicity in the initial velocity field. And if there is than the helicity spectrum would be exactly the same as with the zero initial helicity. The average helicity is largely irrelevant for the turbulence dynamics. Although there is no simple relations between the shell averaged helicity spectrum phase coherence, the same sign of helicity at high wavenumbers, and helicity fluctuations in physical space, but the two are of course related. If the helicity associated phases are numerically randomized the helicity structures in physical space would partially lose their coherence.

the K41 spectrum is a particular consequence. This assertion was also made before in this Section, but now we arrived at it in a slightly different manner.

In **Fig. 10** the plot of total helicity time trace is shown for the case of forced steady state turbulence as was discussed previously. It is peculiar in that from time to time after a few large eddy turnover times the helicity sign abruptly changes. The corresponding spectra

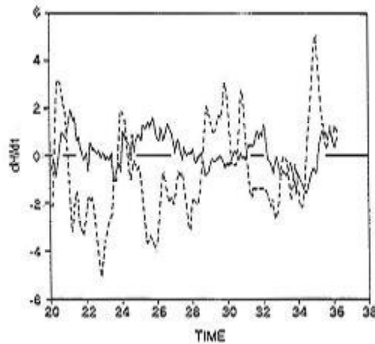


Figure 10: Shows the total H , solid line and dH/dt , dash line, as functions of time normalized by eddy turnover time. They change sign typically at the same time and this signifies phase coherence not only between the phases from the same shell in k -space but between the phases in the high wavenumbers range and the low wavenumbers range. This is because helicity is primarily determined by the high wavenumbers harmonics but its time derivative, the flux, by the low wavenumbers harmonics.

which are not shown here of course change their sign as well while retaining the above phase coherence. $\pm H^s(k) \approx E^s(k)$ in the high wavenumbers range holds in all realizations (representative) and the sign for all high wavenumbers changes one can say almost quasi-periodically and typically simultaneously for all wavenumbers at the same instant of time.⁴¹

⁴¹Some of the past criticism of helical concept, as some authors formulated it, was that the observed helicity is "small". This smallness or largeness of helicity is difficult to quantify and therefore leads to confusion. Indeed, the helicity amplitudes generated by the Navier-Stokes dynamics are small when compared with the mathematically possible as defined by the inequality (3.48). But it is associated phase coherence that is important. Since both in physical space and in conjugate space the helicity fluctuations are of opposite sign this greatly reduces the amplitudes of partially averaged quantities and makes them nearly zero when ensemble averaged. Only the variance of fluctuations in Fourier space remain large. But in physical space the subsequent corollary are **BCC** with velocity and vorticity almost aligned in the constituent cells.

If to think about it the fine tuning of the helicity related phases dynamics is quite impressive. In order for energy to dissipate it should cascade to high wavenumbers where the viscous forces can convert it to heat by molecular friction. But this cascade is not possible unless in Fourier space the helicity phases are not finely tuned in such an extraordinary precise order, so that not to impede with the global balance of energy dissipation. At the same time in physical space **BCC** are formed. Note that in representative turbulence realizations the helical structures in **BCC** though of opposite sign and screening each other still do not totally cancel entirely so that some residual helicity remains. It will become zero in theoretically ideal mirror symmetric situation but only when the ensemble average is done over many realizations and probably very slowly. This is the reason behind the assertions of spontaneous break of mirror symmetry that was to some extent observed in experiment cited previously (e.g., Kholmyansky et al.,2001).

6 Prandtl - Karman Theory of Wall Bounded Turbulence

In real environment turbulence always bounded by walls or free surfaces like an interface between two fluid media, e.g., air and ocean. We shall recapitulate the conceptual aspects of semi-empirical theory of wall bounded turbulence pioneered by Prandtl and Karman. This theory has been the mainstay of theoretical and engineering treatment of BL turbulent flows for nearly a century although as much as in HIT model the full theory of wall bounded turbulence or even its qualitative understanding remain far from completion.

In all turbulent flows the velocity field can be seen as a sum of systematic mean velocity obtained by statistical averaging over time and additional fluctuating velocity component that has the order of magnitude for a scale l equal to changing the mean velocity that occurs for a displacement of order l . It is this fluctuating component of the velocity field flow for which the previous theory was devoted. And as the principle of universality states this turbulence at small enough scales does not depend on the mean flow properties. But only the eddies having the scales quite smaller than the distance from the walls $l(y) \ll y$ will be likely not affected by boundary effects.

First of all let us introduce for reference the Reynolds equation that is a mainstay of theoretical and engineering treatment of wall bounded flows. The description of wall bounded flows start from the Reynolds decomposition of the velocity field on the mean and fluctuating parts:

$$\begin{aligned} \mathbf{V}(\mathbf{r}, t) &= \mathbf{U}(y) + \mathbf{v}(\mathbf{r}, t), \\ \mathbf{U}(y) &= \langle \mathbf{V}(\mathbf{r}, t) \rangle, \\ \langle \mathbf{v}(\mathbf{r}, t) \rangle &= 0, \\ \mathbf{v}(\mathbf{r}, t) &= \{u = v_x; v = v_y; w = v_z\} \\ P &= P_0 + P', \\ P_0 &= \langle P \rangle. \end{aligned} \tag{6.1}$$

If (6.1) is substituted into the Navier-Stokes equations (1.1) and perform the averaging operation we obtain the Reynolds equation for the mean flow $U(y)$:

$$\partial_t U_i + \partial_k (U_i U_k) = -\partial_i P_0 + \partial_k (\nu \partial_k U_i - \langle v_i v_k \rangle). \tag{6.2}$$

The basic term in these equations is the Reynolds stress tensor $\tau_{ik} = -\langle v_i v_k \rangle$ that formally adds to the molecular stress. In (6.2) it is written in a way that emphasizes its usual interpretation as transport due to turbulent "diffusion" implying that part of the mean flow energy is given away to the fluctuating motion and it is this one that is responsible for material transfer and subsequent viscous dissipation by the mechanisms described previously. On the whole the mean flow energy is lost to turbulent motion and ultimately dissipated by molecular viscosity. The Reynolds equations are of course not closed because while the Navier-Stokes equations and the continuity equations are four equations for three velocity components and pressure the Reynolds equations now have a new unknown quantity, the Reynolds stress. The fifth equation for the Reynolds stress can be derived like it was done for the mean velocity in (6.2) but it will depend now on the third order correlator $\langle v_i v_j v_k \rangle$, a new unknown quantity and so forth. Eventually we shall encounter a situation when an equation for the $N - order$ velocity correlation function depends on the $N + 1 - order$ velocity correlation function. Such system can be closed by truncation at a certain order and the consequences will

be discussed in the next section. It should be just mentioned that in wall bounded turbulence closures of any kind result in an especially dire failure of the subsequent theory to account for coherence and CS and hence are totally inadequate as far as the fundamental theory is concerned.

Let us simplify now the Reynolds equation by choosing a channel flow as a model for consideration. Turbulent flow in a flat infinite channel with the walls separated by distance 2Δ is by far the simplest example of flows both in laminar and turbulent regimes. In a laminar regime the channel flow (and similar pipe flow) are given by the well known exact Poiseuille solution of the Navier-Stokes equations. For certain values of the Reynolds number the solution becomes unstable and turbulence ensues.⁴² The well developed turbulent channel flow then despite its deceiving simplicity, nevertheless, on a level of principles show almost all basic features and richness of wall bounded turbulence.

For infinite channel flow Cartesian coordinates $\mathbf{r} = (x, y, z)$ are chosen in such a way that (x, z) are parallel to the walls, with x in the streamwise direction, z in spanwise direction and y normal to the walls. We consider a steady flow with the mean velocity $U(y) = U(2\Delta - y)$ that is time independent and from symmetry considerations a function only of the distance from the walls y . Steady state flow is sustained by a constant pressure gradient along the channel length. The Reynolds equations for the channel flows become quite simplified and look as follows:

$$\begin{aligned}\partial_y \langle uv \rangle &= -\partial_x P + \nu \partial_y^2 U, \\ \partial_y^2 v^2 &= -\partial_y P.\end{aligned}\tag{6.3}$$

The second equation yields $P + \langle v^2 \rangle = P_0(x)$, the pressure at the walls. Also, $\partial_x^2 v^2 = 0$, so that $\partial_x P = \partial_x P_0$. Therefore we can integrate once the first equation (6.3) over y yielding:

$$\langle uv \rangle = -\partial_x P_0(y - \Delta) + \nu \partial_y U,\tag{6.4}$$

⁴²Despite the fact that the laminar flow solutions are the same for a flat channel and pipe the instability of the two flows are two very distinct problems. If the instability of the Poiseuille channel flows is rather simple problem, on the contrary for the pipe flows the problem is much more difficult.

where $\langle uv \rangle$ is the Reynolds stress. It should be noted that due to the channel symmetry $\langle uv \rangle$ at the centerline Δ . The equation (6.4) look simple but it is a deceiving simplicity. Because we still know nothing about the Reynolds stress. Nevertheless, the symmetries of the channel flow allow making some deep conclusions concerning its properties (e.g., Townsend, 1980).

Briefly we develop the reasoning as follows. The pressure gradient term $\partial_x P_0 = \tau_0$ is a friction force that is the same as the stress acting on the channel walls. Let us rewrite (6.4) as follows:

$$\tau = -\langle uv \rangle + \nu \partial_y U = \tau_0(1 - y/\Delta). \quad (6.5)$$

What (6.5) says is that the total stress is a sum of the viscous stress and the Reynolds stress. There are three constants in the flow: τ_0 , Δ and ν . The characteristic Reynolds number is defined as:

$$Re_\tau = \tau^{1/2} \Delta / \nu. \quad (6.6)$$

This definition of the Reynolds number often very convenient for turbulence near the wall and used often together with the conventional definition

$$Re = U_{center} \Delta / \nu, \quad (6.7)$$

where U_{center} is the mean velocity at the channel centerline. It is assumed that everywhere in the flow $\langle uv \rangle \leq 0$, meaning that the mean averaged flow of energy and momentum is from the mean flow to the fluctuating velocity component, than certain deductions can be made concerning the mean flow $U(y)$. Primarily it relates for comparison between the profile of $U(y)$ and the corresponding laminar flow profile, Poiseuille velocity $U(y)_{Lam}$ for the same value of Re_τ , if it assumed that such laminar flow would exist and not destabilize in favor of turbulent flow. Without going into details we reiterate the well known classical conclusion that the turbulent profile is flatter than the laminar, so that:

$$\partial_y U \leq \partial_y U_{Lam}. \quad (6.8)$$

The obvious consequence is that for the same friction force at the walls τ_0 , the total turbulent flow flux, or we can call it the channel flow throughput, is less than the corresponding laminar flow throughput.

In other words for the same throughput the friction at the walls for a turbulent flow is higher than for the laminar. This effect is easy to understand. The fluctuating velocity turbulent eddies invade the mean flow in all directions and in particular towards the wall. This effectively decelerates the mean flow. At the wall the velocity is zero due to the no-slip boundary condition (1.5) and this means that the friction and in consequence dissipation are the largest at the walls proximity. Nearer to the wall the faster is the dissipation of eddies energy due to the molecular viscosity. Thus there are systematic momentum flux and corresponding energy flux in the direction of the wall. Turbulence is essentially a loss of energy in wall bounded flows due to the turbulence induced additional, by comparison with the laminar flows, wall friction. For large Re_τ the energy losses are huge. Decreasing turbulent friction at the walls is the main task for turbulence engineering. And the hope always has been that better understanding of turbulence would help turbulence management.

Since from symmetry the Reynolds stress should be zero at the channel center line, i.e., $\tau(\Delta) = 0$, and at the walls $\tau(0) = \tau(2\Delta) = 0$, it follows that somewhere in the flow there is a location y_{max} at which $\tau(y_{max}) = \tau_{max}$. This rather obvious conclusion will be useful for what follows in Section 10. It is clear that for some distances from the wall $y < \delta$ the viscous stress:

$$| -\nu\partial_y U(y \leq \delta) | \geq | \langle uv \rangle_{y \leq \delta} |. \quad (6.9)$$

This defines the wall viscous sublayer of thickness $y \leq \delta$, which in physical space for wall bounded turbulence is what the viscous subrange $l \leq l_d = k_0^{-1}$ is in the space of scales in HIT turbulence. Naturally the flow in the viscous sublayer is not at all laminar, as it is not in the viscous subrange in HIT turbulence. The turbulent eddies penetrate in the viscous sublayer and there are damped by the nearness to the wall at which the flow freezes because of the no-slip boundary condition (1.5).

Outside of the viscous sublayer $y \gg \delta$ the Reynolds stress is large by comparison with the viscous one:

$$| \langle uv \rangle_{y \gg \delta} | \gg | \nu\partial_y U(y \gg \delta) |. \quad (6.10)$$

It is reasonable to assume that there is a region sufficiently away from the wall where the mean flow is independent of ν . In this region

it is assumed that the mean flow is universal.⁴³ It means that we can write for the velocity profile, in a way similarly to (2.19), the following general scaling relation:

$$U = U_0 + v^* \phi_1(y/\Delta; Re_\tau), \quad (6.11)$$

where the velocities U_0 and v^* are two empirical parameters characterizing the flow. In the limit $Re_\tau \rightarrow \infty$ the empirical function ϕ_1 degenerates, so that $\phi_1(y/\Delta; Re_\tau) \rightarrow \phi(y/\Delta)$. The same reasoning leads us to conclusion that in the universal region:

$$\tau = \langle uv \rangle = \phi_{12}(y/\Delta), \quad (6.12)$$

where ϕ_{12} is another empirical function. As far as the constants U_0 and v^* are concerned we choose U_0 arbitrary but define:

$$v^* = \tau^{1/2}. \quad (6.13)$$

The meaning of (6.13) can be deduced from the following consideration. Let us assume that there is a region in the channel which is, on the one hand, close to the wall although outside the viscous sublayer so that it is, on the other hand, inside the region of well developed turbulence. Because of the second condition the total stress is determined by the Reynolds stress but, because of the first condition, continuity requires that $\tau \approx \tau_0 = \text{constant}$. Therefore from very general considerations it follows that there exists a region of constant (approximately) momentum flux to the wall. Actually this is clear from Eq. (6.5) for $\Delta \ll y$. The velocity v^* is the characteristic fluctuating velocity at the boundary with viscous sublayer $y \approx \delta$ with Re_τ , and is called friction velocity for association with the wall friction.

What can be deduced about the function ϕ_1 in the limit $Re_\tau \rightarrow \infty$ and sufficiently far away from the walls? Let us introduce the eddy viscosity similar to the one in HIT model given by (2.11) but describing the energy dissipation as the result of the momentum flux to the walls. It is qualitatively clear that as the walls are approached the eddies sizes decrease and in this sense this can be seen as a cascade

⁴³The universality hypothesis is very similar to the K41 assumptions of universality in HIT model. But in fact it was first made by Prandtl in 1925.

of energy from larger to smaller eddies, but in physical space. In general we can conveniently imagine that if in the case of homogeneous turbulence the energy flux and dissipation take place in $3D$ space of wavenumbers and in inhomogeneous turbulence these take place in $(3 + 3)D$ space of wavenumbers and coordinates. In conjunction the flux in the generalized configurational space is the energy-momentum flux $\{\epsilon(\mathbf{r}, \tau\mathbf{r})\}$. In the degenerate case of channel flow it becomes the flux of two quantities, $\{\epsilon(y), \tau \approx \text{constant}\}$.

We write like in (2.12) but substituting the distance from the wall y instead of the size of the eddies l_0 . Because the largest eddies cannot exceed in size the distance from the wall.⁴⁴ Evidently from the meaning of eddy viscosity and with reference to (2.13):

$$\langle \epsilon \rangle (y) \approx v(y)_{eddy} (\partial_y U)^2 = v(y)_{eddy} (dU/dy)^2. \quad (6.14)$$

Indeed, $\nu_{eddy} dU/dy$ stands here exactly as the stress as a result of which energy is passed over to a smaller distance from the wall and in this sense dissipated to the smaller scale eddies. This energy is also dissipates through cascading to the respective dissipation scales at every y (dissipation scales now themselves depend on y). At the same time from the K41 and (2.11) it follows that for the largest eddies of order l_0 :

$$\nu_{eddy} \approx v_0 l_0 = \langle \epsilon(y) \rangle^{1/3} y^{4/3}, \quad (6.15)$$

since now $l_0 = l_0(y) \approx y$. Comparing we obtain:

$$\nu_{eddy} \approx y^2 dU/dy. \quad (6.16)$$

Let us calculate now $\langle \epsilon \rangle$ from different and independent considerations. Consider the general expression for the momentum flux (1.17) and apply for the channel flow geometry, using the Reynolds decomposition (6.1) and averaging over time. We have evidently:

$$\begin{aligned} \langle j_k^E \rangle &= - \langle v_k (1/2 v_i^2 + P) - 2\nu V_i e_{ik} \rangle = j_y^E = \\ &= - \langle \{(P_0 + P') + (\mathbf{v} + \mathbf{U})^2 / 2\} v_y \rangle = \end{aligned}$$

⁴⁴This presumes in the spirit of K41 theory that the eddies are isotropic. In reality they are not even in BigBox turbulence with ergodic boundary conditions. And definitely not in the presence of the mean flow. But apparently the averaging and scaling laws generality prevail over the anisotropy often, but not at all always.

$$\begin{aligned}
 = & \langle \{P' + (u + U)^2 + v^2 + w^2/2\}v \rangle = -U \langle uv \rangle - \langle P'v \rangle + \\
 & -(u^2v + v^3 + w^2v)/2.
 \end{aligned}
 \tag{6.17}$$

Empirically $U \gg \{u, v, w\}$, so that the last term in brackets can be neglected. Also $P' \approx v^3$. So that finally with good accuracy:

$$j_y^E = -U \langle uv \rangle = U\tau.
 \tag{6.18}$$

In the region where the Reynolds stress $\tau \approx \tau_0 = \text{constant}$ we obtain the classical expression:

$$j_y^E = U\tau_0.
 \tag{6.19}$$

The energy flux should diminishes as the wall is approached because the energy dissipates at every distance from the wall. It is useful to have a picture in mind that at every y there is also the Kolmogorov cascade in the space of scales, in x, z plane, with subsequent viscous dissipation. The difference is that the scales have become functions of y and this y dependence is responsible and tantamount to the energy and momentum flux in the wall direction.

Evidently the dissipation rate as a measure of change of the energy flux as one approaches the wall with progressively smaller eddies is as follows:

$$\langle \epsilon(y) \rangle = \partial_y j_y^E = - \langle uv \rangle dU/dy.
 \tag{6.20}$$

Note that it is assumed:

$$\langle uv \rangle dU/dy \gg U(y)d \langle uv \rangle /dy.
 \tag{6.21}$$

This condition actually defines the required approximate constancy of the Reynolds stress. Together the relations (6.20), (6.16) and (6.14) yield:

$$dU/dy = \tau_0^{1/2}/\kappa y,
 \tag{6.22}$$

where the empirical von Karman constant κ was introduced. It is clear that all the previous considerations were losing constant factors, for instance and probably most importantly a constant in front of the K41 expression (6.15) and κ means to absorb for the losses. Integrating we obtain for the mean velocity profile the most important expression, except the K41 spectrum standing in the same tier of values, in the history of turbulence research:

$$U(y) = \tau_0^{1/2}/\kappa(\ln y + C),
 \tag{6.23}$$

where C is an unknown constant to be determined semi-empirically. This can be done as follows. Consider the viscous sublayer. The mean flow velocity there is dominated by viscous stress at the walls and therefore one writes:

$$\tau_0 = \nu dU(y \leq \delta)/dy, \quad (6.24)$$

so that:

$$U(y \leq \delta) \approx \tau_0 y / \nu = v^* y / \delta. \quad (6.25)$$

At $y \sim \delta$ it should be that the mean velocity matches the friction velocity, i.e. $U(\delta) \approx v^*$. Then the only way to satisfy this matching of the velocities would be to choose C from the condition:

$$U(\delta) = \tau_0^{1/2} / \kappa (\ln \delta + C) \approx v^* = \tau_0^{1/2}. \quad (6.26)$$

Experimentally $\kappa \approx 0.4$. Eventually we get the following law:

$$U(y)/v^* = U^+(y) = 2.5(\ln y / 0.13\delta) = (2.5 \ln y^+ + 5.1), \quad (6.27)$$

where $y^+ = y/\delta = y/(\nu/v^*)$ is the distance from the wall in wall units measured by the thickness of the viscous sublayer. In these units the Reynolds number (6.6) becomes as follows:

$$Re_\tau = \tau_0^{1/2} \Delta / \nu = v^* \Delta / \nu = \Delta / \delta. \quad (6.28)$$

The law for the energy dissipation, or at the same for turbulence production (6.20) becomes as follows:

$$\langle \epsilon(y) \rangle \approx \tau_y^{1/2} / y. \quad (6.29)$$

It shows that the closer to the wall the more intensive is the energy dissipation due to viscous forces. This cannot continue into the region in close proximity with the wall of course. But close to the viscous sublayer, actually in the *buffer zone* the conditions that led to the law of the wall (6.27) fail. It was established experimentally that the dissipation and therefore turbulence production reach maximum at a fixed distance from the wall $y^+ \approx 13$ that seems independent of the Reynolds number. It seems established that the most intensive turbulence activity takes place well inside the so-called *buffer zone* that connects the regions with the universal logarithmic profile and

the viscous sublayer. The similarity between (6.29) in physical space and (3.46) in conjugate space in HIT is obvious. But again let us emphasize that while in HIT it was the constancy of the energy flux in Fourier space that was the basic assumption, in wall bounded flows it is the momentum flux constancy in physical space that is playing similar role.

The law of the wall (6.27) is shown in **Figs. 11** from DNS is typical for all DNS of turbulent channel flow with moderate $Re_\tau = 125$ and good resolution.⁴⁵ It is also confirmed in hundreds of laboratory measurements. Although for very high Reynolds number flows the values of the empirical constants may somewhat differ from the ones for moderate Reynolds number flows. Nevertheless, the general self similarity principles clearly and obviously work in this case. The above derivation is one of many that lead to the same logarithmic law. The one that was chosen here is to emphasize that there is definite relation and affinity between the self similarity in HIT and self similarity in a wall bounded turbulence. If there was no K41 spectrum there would be no logarithmic law (6.27). From general considerations it was mentioned before that the Reynolds stress must pass through the maximum at a certain distance from the wall. Let us find its location. Using Eq. (6.5) and differentiating over y we obtain (Sirovich, et.al., 1991):

$$\nu d^2U/dy^2|_{y=y_{max}} = \tau_0/\Delta, \quad (6.30)$$

or in dimensionless units:

$$Re_\tau d^2U/dy^{+2}|_{y^+_{max}} = 1. \quad (6.31)$$

In the universal range therefore we obtain the following asymptotic expression for the position of the maximum of the Reynolds stress:

$$y^+_{max} \approx \sqrt{Re_\tau/\kappa}. \quad (6.32)$$

This is an interesting expression that tells us that even though the universal range and the logarithmic law of the wall seem to require

⁴⁵The DNS of turbulent channel flows are usually carried out with no-slip boundary conditions at the walls at $y = 0$ and $y = 2\Delta$, periodic boundary conditions in x, z . The velocity field is represented as the usual Fourier decomposition in x, z and as expansion in Chebyshev polynomials in $y - direction$.

that the momentum flux is constant, but nevertheless this is not the case and the momentum flux reaches a maximum and declines with the further approach to the wall and this is while the wall of the wall remain intact. In fact only the condition (6.21) should be true and this is always the case in the logarithmic profile range. Note that although in the wall units the position of y^+_{max} seems to get farther from the walls with the growth of Re_τ in physical space this position approaches the wall as $y_{max} \propto Re - \tau^{-1/2}$.

In **Fig. 11** one can see three distinct regions in the flow. Away from the wall, at about $y^+ > 30$ the fit with logarithmic profile, with constants close enough to the ones in (6.27), is relatively good but it could be also approximated by a power law function with a small exponent. Near to the wall $y^+ < 5$ the profile is (6.25) and in the range approximately defined as $5 < y^+ < 30$ and that is the *buffer zone* where the profile cannot be determined by the above reasoning. It is natural to think that this is the region where the nonlinear coupling and in consequence the Reynolds stress are of the same order as the viscous stress and therefore both are important. The Reynolds numbers are too low in these DNS runs to make definitive conclusions on the exact mean velocity profile or on the boundary between the *buffer zone* and the universal range.

There are no theoretical arguments in the above theory for determining the profile in the buffer zone or the distance from the wall where it merges the logarithmic profile and this data should be inferred from measurements. It should be noted however that despite the undeniable beauty of the above semi-empirical theory there are many aspects that are what they are-empirical. For instance the length of the buffer zone is usually considered as cast in stone, in the sense that the value $y^+ = 30$ at which it allegedly merges with the logarithmic profile region is quoted as universal and Re_τ independent. However some authors give other numbers and assert that, ostensibly for high Reynolds numbers flows, the boundary between the buffer zone and logarithmic law is at $y^+ = 60$ and even farther at $y^+ = 70$ from the wall (e.g., Hinze 1975; Schlichting, 1968). In other words a weak dependence from the Reynolds number cannot be ruled out. For comparison we show an example of experimentally obtained mean profile in **Fig. 12**. It is clear that up to $y^+ \approx 50$ the agreement with logarithmic law is poor.

The logarithmic law of the wall is of great importance for engineering turbulence since it allows making universal predictions for the flows with arbitrary high Reynolds numbers, for which the direct measurements are problematic since the relevant physical distances from the wall become increasingly small for very high Reynolds numbers, if the empirical constants are firmly established from the measurements at moderate Reynolds numbers. The most basic engineering parameter for wall bounded turbulence is the friction coefficient that defines energy losses due to turbulence. It is reminded that the most intensive dissipation occurs in the shallow layer near to the walls where the conversion of the mean flow energy into turbulence and the subsequent dissipation reaches the maximum, even though the flux of energy to the walls is furnished from the whole bulk flow. But it is the slowing down of the bulk flow that is the measure of the total energy losses near to wall. This is quite similar to what happens in HIT turbulence. The energy is primarily dissipated by the small scale eddies of order l_d but the energy that lost is brought by the energy flux furnished by the large scale motion that determines the total energy of the flow.

It was mentioned before and is quite obvious that the losses due to turbulence are much higher than in the laminar flow, if they could exist for the same Reynolds numbers. The fluctuating eddies emerging from the mean flow and then randomly penetrating it in all directions are the obstacles serving to decelerate the flow. This is briefly how it happens more quantitatively. Let us consider the mean flow velocity at the channel center assuming that approximately it still follows the law (6.27). We have in physical units:

$$U_{center} = v^* / \kappa \ln \Delta v^* / v, \quad (6.33)$$

where the centerline velocity U_{center} is actually the throughput of the channel, i.e., the fluid volume flowing per unit time through the cross section of the channel divided by the area of the cross section. This flow is supported by the constant pressure gradient along the channel length $d \langle P \rangle / dx = constant + A$. This pressure drop acts to compensate the walls friction per unit area that is equal to $\tau_0 = v^{*2}$. Therefore $d \langle P \rangle / dx = v^{*2} / \Delta$. Hence we obtain the relation:

$$U_{center} = (\kappa)^{-1} (\Delta A)^{1/2} \ln \Delta / \nu (\Delta A), \quad (6.34)$$

that connects the throughput with the pressure drop. Introducing the dimensionless friction coefficient:

$$C_f = 2\Delta A/U_{center}^2, \quad (6.35)$$

it is easy to derive the following classical parametric equation for it (e.g., Landau and Lifshitz, 1979):

$$1/\sqrt{C_f} = 0.88 \ln Re \sqrt{C_f} - 0.85, \quad (6.36)$$

where the constant factors are in correspondence with the ones in the logarithmic law of the wall (6.27). For comparison the friction coefficient for the laminar Poiseuille flow is $C_{f,lam} = 12/Re$. Although it is not immediately seen from the comparison of C_f and $C_{f,lam}$ but in fact the latter falls down for high Reynolds numbers much faster than the former. In some approximation the turbulent friction coefficient can be shown to have the asymptotic $C_f \propto Re^{-1/2}$ in a wide range of large Re .⁴⁶ On a big industrial scale the turbulent drag means in practice huge energy losses in any and every system involving fluid flows.

Is it possible to reduce the turbulent drag? Enormous work has been done towards this goal and remarkable empirical successes were achieved in the past that are evidenced by the modern flying machines and the ships. But without clear understanding of the fundamentals of turbulence beyond phenomenological theories the systematic turbulent management remains an elusive goal.

Note that since the friction is determined by two factors, the viscous "skin" friction and turbulent Reynolds stress and in the above approximations we convinced ourselves that the latter is much larger in most of the flow it appears that the only way to reduce turbulent drag is to reduce the turbulent stress $\langle uv \rangle$ and the question is how.

Despite the beauty of the above theory real wall bounded turbulence is much more complicated and the knowledge of mean velocity profile and drag coefficient are not enough.

⁴⁶While the friction coefficient tends to zero the total energy lost of course grows with the growth of Re and for turbulent flows much faster than for the laminar ones by virtue of much larger C_f .

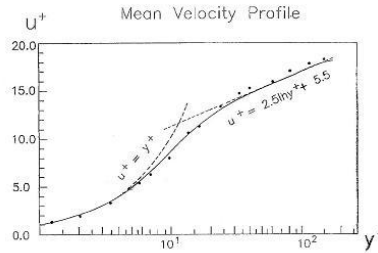


Figure 11: Shows the mean velocity profile in a turbulent channel flow from DNS by Sirovich, et. al. (1991). The Reynolds numbers are: $Re_\tau = 125$ and $Re \approx 1800$. The profile looks qualitatively similar to the ones measured in laboratory experiments. There is confidence in certain features, such as the logarithmic profile, at least asymptotically for large Re_τ , sufficiently far away from the walls, the presence of the buffer zone and viscous sublayer.

If turbulence is a quasi-Gaussian phenomenon as is implied in K41 theory and in Prandtl - Karman theory for wall bounded flows then generally this is not possible, except some particular situations that have been primarily empirically discovered during the long history of trials and errors in turbulent research. Moreover these situations are clearly out of scope of Prandtl - Karman theory.

The main reason for further research, except scientific curiosity, is the ever present desire to control turbulent drag. Usually it is necessary to diminish drag, but sometimes we want to increase it. The reason for the latter is that in many applications it is beneficial to increase the transport of admixtures to the walls and heat transfer from the walls. The increase of turbulent mixing and transport is usually linked with the increase of drag.

The conjecture that is made here is that the near to wall turbulent region is the location of intensive **BCC**, as much as the high wave number region is such in HIT. This conjecture for wall bounded flows was made in Levich (1996), although the helical nature of wall bounded turbulence and geophysical structures was asserted earlier in Tsinober and Levich (1983), Levich, Tsinober 1984 and Levich

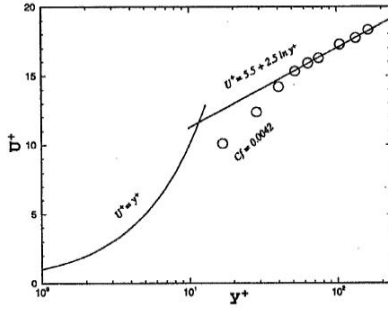


Figure 12: From Rajaei, et.al. (1994). Plots experimental mean velocity profile obtained in a channel flow with the $Re = U_{center}\Delta/\nu \approx 10.000$. The deviation from logarithmic profile is clear till $y^+ < 50$. C_f is the friction coefficient (6.35).

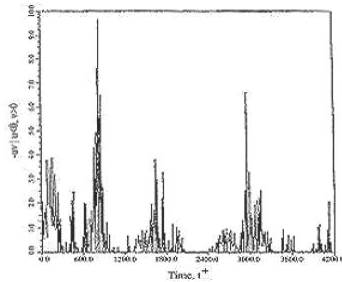


Figure 13: Shows the typical time (in wall units) trace from DNS by Sirovich, et.al. (1991) of the Reynolds stress $-uv > 0$ in the quadrant $u < 0$ and $v > 0$. The time is normalized by the eddy turnover time and the acute intermittence of the signal, intense peaks separated by the quiescent intervals, is evident (compare with Fig. 7). The time relates to typical turnover time expressed in universal wall units. The peaks of activity are usually attributed to so-called bursting and sweeping events (see Section 10 below).

Tzvetkov (1984, 1985). Unfortunately no analysis similar to that of Mininni, et.al. (2008a, 2008b) for HIT has been accomplished for turbulence near to wall, even in the simplest case like turbulent channel flow. This issue will be further addressed below in Section 10.

In **Fig. 13** the time trace is shown (Sirovich, et.al., 1991) of the stress $u(t)v(t)$ for a location near to the wall in the *buffer zone*. The time trace was generated by DNS with the same parameters as for the one in **Fig. 11**. It is sampled out for simplicity in such a way that the stress is negative everywhere. It can be clearly seen that the time trace is strongly intermittent with strong peaks of intensity separated by rather long periods of relative quiescence. As much as intermittency is typical for the high wavenumbers harmonics of HIT turbulence so it is in the proximity to the wall in wall bounded flows. It was discovered in the last 50 years of observations that near to wall flow region is full of relatively long lived and having definite shape correlated vorticity sub-domains called coherent structures (CS). As was explained in Foreword CS have been so far described purely on a level of visualization, as sub-domains of seemingly correlated vorticity. It is my personal belief that these CS are similar to the intense vorticity bands in **Fig. 2** for HIT. But a variety of such structures in near to wall region that are observed and talked about appears very rich compared to HIT where researchers are reluctant at times to admit coherence at all. These are pin vortices, rolls, streaks, horse shoe vortices, etc. Unfortunately no much sense has been made till now with all this multitude of vorticity shapes, little is understood of their origin and the role they play. However it is absolutely clear to everyone that they are dynamically important and in fact are turbulence. The Prandtl - Karman theory is as incomplete as K41 theory by not accounting the coherent manifestations and in principle not being capable of addressing them.

7 Dynamical Theory 1: Perturbation Theory and RNG Analysis in Asymptotic Limit of Low Wave Numbers and Frequencies

There were numerous attempts to find analytical approach to K41 theory from the first principles, i.e., from the Navier-Stokes equations. All the approaches were based on perturbation theories in one

way or another and closures; the perturbation theories methods were primarily borrowed from certain successful applications to non-linear quantum field theories and closures from unsuccessful applications in statistical mechanics of strongly interacting gases. The emphasis was on the K41 spectrum, because for intermittency and coherent structures except for phenomenological models no theoretical approach was ever invented. Besides the way many were thinking of turbulence remained linear in the sense that intermittency and CS were seen as corrections laid over the basically healthy K41 theory. Still other authors noted that the energy spectrum is just one of the basic quantities describing turbulence. It does not carry useful information on extreme deviations of turbulent fields from the mean and does not contain any information on CS. Nevertheless, one should start from something and the researchers focused on the K41 spectrum as the only quantitative result at the time and it took good 40-50 years before the attempts to devise yet another different perturbation theory from the Navier-Stokes equations finally came to end. Nevertheless, the K41 spectrum itself remains a foundation for the models of turbulence in which the high wavenumber properties of turbulence are treated in an averaged manner as eddy viscosity for the low wavenumbers turbulence.⁴⁷

Even though the closures and perturbation theories when directly applied fail to reveal the real structure of turbulence important conclusions can be drawn from the details of how and why they fail. Furthermore as will be shown in the next section a particular perturbation theory that is applied together with certain specific assumptions on the structure of turbulence reveals important quantitative

⁴⁷The turbulence models are many and differ from each other by particular ways of formulating the eddy viscosity. They are of course all empirical and contain phenomenological constants that are fit to satisfy the experimental data. The role of extreme deviations from the mean at high wavenumbers is not discussed and such deviations play no role in these models. It is surprising that in many situations and applications the models do work well despite the fact that they are all based on wrong physics. So that one may think that the "linear" thinking may be correct and intermittency though plays obviously the role for high order statistics does not influence the large scales of turbulence and this sense can be disregarded. This is a simplistic position because in many other applications and most importantly in geophysics the models don't work at all. Why and where the models do or don't work is a very important question. It will be in a limited way discussed in what follows.

properties of this structure, e.g., the fractal dimension of the sub-domain occupied by **BCC**.

All these schemes whatever name they carried with them were a perturbation theory in one way or another. And since the only intrinsic parameter in the Navier-Stokes equations is Re it would serve as the perturbation theory expansion parameter. And since it is large the problem was how to obtain converging series from expansion in powers of a large parameter. In reality of course all the perturbation theories were based on the assumption of K41 scaling and contained certain basic assumptions that despite the elaborate mathematical equilibristics and complexity made the perturbation theories results trivial. However certainly a few things were understood that have had lasting importance for understanding the structure of turbulence, even though the complete theories and methods themselves were doomed to oblivion.

Let us start again from formulating the general scaling invariance rules for the Navier-Stokes equations (1.1):

$$\begin{aligned}
 \mathbf{r} &\rightarrow \lambda \mathbf{r}, \\
 t &\rightarrow \lambda^z t, \\
 \mathbf{v} &\rightarrow \lambda^{1-z} \mathbf{v}, \\
 \mathbf{F} &\rightarrow \lambda^{1-2z} \mathbf{F}, \\
 \nu &\rightarrow \lambda^{2-z} \nu.
 \end{aligned}
 \tag{7.1}$$

It can be shown that the exponent $z = 2/3$ corresponds to all K41 theory predictions if the external forcing \mathbf{F} is chosen as a scaling function with the exponent:

$$[\mathbf{F}] = 1 - 2z = -1/3,
 \tag{7.2}$$

where the symbolic square brackets will from now on mean the scaling exponent of a variable inside them, unless specifically indicated otherwise. Note that the fact that an equation is scale invariant with a particular choice of sources, i.e., the forcing in the Navier-Stokes equation, does not mean that there is a real stable scaling solution. Or it can unstable, or there may be a family of solutions and the system flows from one unstable solution to another, which most probably

happens with the turbulent flow field. On the other hand the velocity correlation functions can attain stable scaling form compatible with the general set of scaling transformations. Since we are concerned with the velocity correlation functions rather than the velocity field itself it is natural to choose the source as a random force also defined through the correlation functions. And since the universality is always assumed it is possible to choose it to be a Gaussian random force that we define in Fourier space as follows:

$$\langle F_i(\mathbf{k}, f) F_j(\mathbf{k}', f) \rangle = (\delta_{ij} - k_i k_j / k^2) \Phi(k, f) \delta(\mathbf{k} + \mathbf{k}') \delta(f + f'), \quad (7.3)$$

where the forcing Fourier transform $F_i(k, f)$ is defined regularly and: Since F_i is Gaussian it is fully determined by the correlation function and since the forcing is Gaussian all higher order correlation functions are factorized as products of Φ , i.e., essentially:

$$\langle F^n \rangle \propto \langle \Phi \rangle^{n/2}, \quad (7.4)$$

where the unneeded at this point tensor subscripts were omitted for simplicity. The basic condition for the external source should be as follows:

$$\Phi(k > k_0, f) \rightarrow 0. \quad (7.5)$$

The external force should not be forming the turbulence flow at $k > k_0$ but this should be created by the nonlinear coupling and viscosity, i.e., by the natural cascade dynamics.

Let us consider the Navier-Stokes equation in Fourier space for the velocity Fourier image (3.4):

$$v_i(\mathbf{k}, f) = F_i(\mathbf{k}, f) G(\mathbf{k}, f) - 1/2(i\lambda_0) G(\mathbf{k}, f) P_{ijk} \int d^D q d f' v_j(\mathbf{q}, f') v_s(\mathbf{k}-\mathbf{q}, f=f'), \quad (7.6)$$

where $\lambda = 1$ is introduced for the purposes of generating the perturbation expansion and $D = 3$.

$$G_0(\mathbf{k}, f) = (-if + \nu k^2)^{-1}, \quad (7.7)$$

is the zeroth order (bare) Green function and:

$$P_{ijs}(\mathbf{k}) = (\delta_{ij} - k_i k_j / k^2) k_s + (\delta_{is} - k_i k_s / k^2) k_j. \quad (7.8)$$

Let assume now that instead of having $\Phi(k)$ defined properly as in (7.5) we stir fluid with a random Gaussian force with a scaling power law correlation function stretching for all wavenumbers in the interval $k_0 \ll k \ll k_d$:

$$\Phi(k, f) = A/k^y. \quad (7.9)$$

What one can argue in this case is that the universality should be insensitive to the nature of forcing and the natural solution will be formed anyway. But this is wrong. Because if the forcing (7.9) pumps too much energy into the high wavenumber velocity harmonics the energy spectrum may be a less singular than K41 in the limit of low wavenumbers and flatter than K41 in the limit of high wavenumbers in the inertial range. That is to say that the energy spectrum would be $E(k) \propto k^{-x}$ with $x < 5/3$ and hence the excess of energy by comparison with K41 with $x = 5/3$. It all depends on the value of exponent in (7.9) and the corresponding solution of the Navier-Stokes equation. Such values of $y > y_{critical}$ that indeed lead to $E(k) \propto k^{-x}$ with $x < 5/3$ define a model of randomly stirred fluids (RSF) to distinguish it from the genuine HIT model. It would seem that $x = 5/3$ corresponds to $y_{critical}$ above which the turbulent regime does not realize. Then if we could solve the RSF model we could determine *critical* With the choice of forcing (7.9) the Eqs. (7.6) are naturally also invariant to the set of scaling transformations similar to (7.1) but in Fourier space. To avoid confusion it should be remembered that the rescaling transformation $\mathbf{r} \rightarrow \mathbf{r}$ corresponds to the inverse rescaling in conjugate space, i.e., $\mathbf{k} \rightarrow \lambda^{-1}\mathbf{k}$.

Now we note that the choice of $y = D = 3$ in (7.9) corresponds in physical space to the scaling (7.2); this can be checked by a direct power counting. And equipped with all the above reasoning we may conclude that K41 might be indeed a solution of (7.6)-(7.9) for a particular choice of the exponent y i.e., $\Phi(k, f) = y = D = 3$. But of course it is still necessary to prove that the molecular viscosity is truly renormalized by the nonlinear coupling to become the eddy viscosity (2.11), as is demanded by the last of the scaling transformations (7.1). If this is proved than the velocity field correlation function and the energy spectrum indeed would be K41. This proof was given in a classical paper of DeDominicis and Martin. (1979) in which the authors formulated for the first time the application of renormalization group theory (RNG) for randomly stirred fluids in

the asymptotic limit of low wavenumbers and frequencies $(k, f) \rightarrow 0$.

The RNG yields a converging perturbation theory for RSF model to all orders for the renormalization of molecular viscosity that “dresses up” into eddy viscosity familiar from the K41 theory (see the relation (3.33) for instance) and the asymptotic power law for the correlation function of $v_i(\mathbf{k}, f)$ in the limit of $(k, f) \rightarrow 0$.

The RNG for this model shows how the nonlinear coupling renormalizes, “dresses up” the bare molecular viscosity ν so that it becomes instead of a constant the eddy viscosity scaling function of wavenumber in accordance with the rule (7.1). As one says ν flows in the parameter space and reaches a fixed point. For a particular case of $y = D$ the fixed point is $z = 2/3$ and in consequence the asymptotic solution for the energy spectrum in the limit $(k, f) \rightarrow 0$ is indeed the K41 spectrum. It should be again emphasized that this solution is not really a solution for turbulent flow but for RSF with $y = y_{critical}$ as was explained above. Nonetheless, this is a fairly remarkable result in its own right in retrospect and this should be clearly explained.⁴⁸

Any perturbation theory is developed around the zeroth order solution for the velocity field:

$$v_i^{(0)}(\mathbf{k}, f) = G^0(\mathbf{k}, f)F_i(\mathbf{k}, f). \quad (7.10)$$

This zeroth order solution is what the velocity of fluid would have been if there was no proportional to λ_0 nonlinear coupling. Of course it does not have sense since the whole gist of the Navier-Stokes equations is in nonlinear coupling that generates velocity harmonics and universal velocity field statistical correlators independent of F_i . Hence the perturbation theory should be built in powers of λ_0 .

⁴⁸There are thousands of subsequent papers and books on application of RNG to turbulence. It was considered a breakthrough by some for a period of time for the reasons best left to historians of science to analyze. The passionate expectations coupled with handsome research grants were so high that as a matter of curiosity even such mundane objects as internal combustion engines and likewise devices were treated by means of field theoretical methods of RNG. For a good review of the methods I can refer to McComb (1985). Of course RNG does not even approach properly the theory of turbulence and can be delegated to many other semi-phenomenological approaches that by themselves fail to capture what is most essential in turbulence - the intrinsic coherence. However proper application of RNG in conjunction with far reaching assumptions on the structure of high wavenumbers turbulence allows important calculations to be made as will be demonstrated below.

This is done by substituting the zeroth order solution into the coupling term in (7.6) to generate the next term approximation $v_i^{(1)}(\mathbf{k}, f)$ proportional to λ_0 and then substitute $v_i^{(1)}(\mathbf{k}, f)$ into the coupling term to obtain the next approximation $v_i^{(2)}(\mathbf{k}, f)$ proportional to λ_0^2 and iterate the procedure ad infinitum. After that the statistical averaging over the Gaussian force (7.5) is made. While developing the perturbation theory the first task is to identify the dimensionless parameter of expansion. It was understood by DeDominicis and Martin (1979) that with the choice of forcing (7.9) this parameter is:

$$\lambda = \lambda_0(A/\nu^3)^{1/2}. \quad (7.11)$$

Since there is only one dimensionless parameter in the Navier-Stokes equation, the Reynolds number, λ is just proportional to the Re . In other words the expansion parameter is large and the most interesting case is $Re \rightarrow \infty$. The expansion in powers of infinite parameter should not necessarily deter from the perturbation theory. Such situations were encountered in many problems of field theories and statistical mechanics and sometimes the series can be either regularized or summed up, if one is lucky to end up with finite results. The so-called renormalized perturbation theory was formulated for the Navier-Stokes equations with a general type random force by Wyld (1961). The renormalized perturbation theory reformulates all the bare perturbation series in powers of the exact summed up quantities, but these remain infinite series that cannot be summed up unless some drastic assumptions on the structure of velocity are made. To be more explicit the perturbation theory will renormalizes the Green function, the forcing and the coupling itself as follows:

$$\begin{aligned} G^0 &\rightarrow G^{ren} = (-if + \nu k^2 + \sum(\mathbf{k}, f))^{-1}, \\ \Phi &\rightarrow \Phi^{ren}, \\ \lambda &\rightarrow \lambda^{ren}(\mathbf{k}, f), \end{aligned} \quad (7.12)$$

$$\langle v_i(\mathbf{k}, f)v_j(\mathbf{k}, f) \rangle = |G^{ren}(k, f)|^2 \Phi^{ren}(\delta(ij) - k_i k_j / k^2) \delta(\mathbf{k} + \mathbf{k}') \delta(f + f'),$$

where the self-energy $\sum(\mathbf{k}, f)$ is the renormalized viscosity term similar to the phenomenological eddy viscosity. Similar expressions are obtained for all orders of the velocity correlation functions, which

we will designate symbolically as $\langle v^{(n)} \rangle$, the point is that now, all the quantities in (7.12) are incomprehensible functionals, infinite expansions in powers of $\langle v^{(n)} \rangle$. That is:

$$\begin{aligned} G^{ren} &= G^{ren}(\mathbf{k}, \mathbf{f}; \langle v^{(n)} \rangle, \langle v^{(n-1)} \rangle, \langle v^{(n-2)} \rangle \dots \langle v^{(2)} \rangle; \Phi^{ren}; \lambda^{ren}), \\ \Phi^{ren} &= \Phi^{ren}(\mathbf{k}, \mathbf{f}; \langle v^{(n)} \rangle, \langle v^{(n-1)} \rangle, \langle v^{(n-2)} \rangle \dots \langle v^{(2)} \rangle; \lambda^{ren}; G^{ren}), \\ \lambda^{ren} &= \lambda^{ren}(\mathbf{k}, \mathbf{f}; \langle v^{(n)} \rangle, \langle v^{(n-1)} \rangle, \langle v^{(n-2)} \rangle \dots \langle v^{(2)} \rangle; G^{ren}; \Phi^{ren};), \\ &\langle v^{(n)} \rangle \asymp V(n)(\mathbf{k}, \mathbf{f}; \langle v^{(n-1)} \rangle, \langle v^{(n-2)} \rangle \dots \langle v^{(2)} \rangle; G^{ren}; \Phi^{ren}; \lambda^{ren}), \end{aligned} \tag{7.13}$$

with $n \rightarrow \infty$ and each equation being effectively an infinite expansion in powers of λ^{ren} . The only way to deal with this totally useless system is to break the infinite chain of equations at some finite n' . But the assumptions go even further. To solve the system (7.13) it should be assumed that all order velocity correlation functions $\langle v^{(n)} \rangle$ can be factorized as products of powers of the square correlation functions $\langle v^{(2)} \rangle$. Such procedures are also employed in certain statistical mechanics problems with strong interactions, e.g., dense systems, or the attempts to derive the Navier-Stokes equations from the first principles. In all these cases the explicit or implicit assumptions are made that are equivalent to neglecting the strongly nonlinear nature of interactions so that for all practical purposes they are treated as weakly interacting systems, e.g., rarified gases and by equations similar to Boltzman equation for rarified gases.⁴⁹ Essentially the same happens for (7.13). In fact the inviscid Euler part of the Navier-Stokes equations is of course Hamiltonian (Arnold, 1974), although it is not seen readily. However when made explicit in one way or another the closure of (7.13) results in Boltzman like equations (Levich, 1981). All other forms of closures can yield only equivalent results.^o The result

⁴⁹What is lost in all these treatments is coherence due to strong interactions. In the cases like say superfluidity, for instance, the coherence is possible to recover in the perturbation analysis for the reason that there is one or another small parameter in the system, e. g., a small number of particles passing over to superfluid state in non-ideal Bose gases. This makes the theory renormalizable. In turbulence of fluids there are no parameters except *Re*. An original idea is to seek solution of the Navier-Stokes equations was applied with dexterity for nonlinear analysis of stability of Beltrami flows (Libin, 2008), with $Re^{-1/2}$ as small parameter.

for all of them is exactly the same: K41 spectrum and no intermittency, no coherence, no structures. And this is easy to understand. Because the instant we make a closure and express the high order velocity correlation functions as products of powers of the pair correlation functions we throw a child from the bath together with water because we assume a quasi-Gaussian approximation for the velocity field. The usual justification for this was that experimentally the velocity field at one point is nearly Gaussian. And at two points is not so far from the Gaussian and so forth. But we know that the high wavenumber velocity harmonics are strongly non-Gaussian and if the random phase approximation is assumed to the helicity related phases (the angle between $Re\mathbf{v}(\mathbf{k}, f)$ and $Im\mathbf{v}(\mathbf{k}, f)$, see Section 5) this would leave to severe contradictions.⁵⁰

The RNG is generally speaking different from the usual perturbation theories and closures in that it does not make *a priori* drastic assumptions of locality and effectively weak interaction and on the contrary was successful, for instance in second order phase transition theories, in elucidation of strongly interacting coherent fluctuations of order parameters. This is why RNG initially generated hopes among physicists who tried to roll in with their success in one enigmatic field of research into another. Unfortunately turbulence is vastly more complex problem and the success did not repeat. Nevertheless, certain important lessons could be learned, as well as lessons were drawn from the perturbation theories and closures, and these can be all used to make another step towards understanding of turbulence as dynamical system.

The RNG for practical applications also needs the perturbation theory but different in nature. Also, by nature the RNG analysis is asymptotic. It deals either with $(k, f) \rightarrow 0$ limit or the opposite $(k, f) \rightarrow \infty$ limit, traditionally called respectively "infrared" and "ultraviolet" limits by association with the quantum field theories.

Let us consider briefly how it works for the RSF model (7.9) in the

⁵⁰If purely Gaussian approximation is assumed then the triple order velocity correlation functions that are responsible for the energy transfer to high wavenumbers would be identically zero. Thus certain rudimentary phase coherence is always present (Batchelor, 1953). But in the quasi-Gaussian approximation only even order correlation function are factorized as products of the pair correlation function and triple order correlation function is then automatically also expressed through the same (e.g., Monin and Yaglom 1975).

limit $(k, f) \rightarrow 0$. There are several steps involved in RNG application to dynamical systems. Also, there are different versions of RNG more or less equivalent to each other. Below the exposition will be a little bit different, a mixture of classical steps and some purely technical innovation, to prepare readers to the next section in which an altogether novel approach will be developed.

1. First we define the wavenumbers space, k -space, for the problem $k_d \geq k > 0$. It is important to remember that there is no low wavenumbers cutoff since the infrared asymptotic limit $(k, f \rightarrow 0$ is considered. Consider a shell space in the wavenumbers space:

$$k_d \geq q > k_d e^l - 1, \tag{7.14}$$

where l is an infinitesimal scaling parameter introduced for the fractal analysis in physical space in the previous section, but now the rescaling is in the direction of smaller wavenumbers or bigger scales, hence l reverses but reverses it again since we are considering now the scaling in conjugate space. Let us formally split the nonlinear term in (7.6) as follows:

$$J\{\mathbf{v}\}_i = J_i^<\{\mathbf{v}\} + J_i^>\{\mathbf{v}\}, \tag{7.15}$$

where:

$$J_i^>\{\mathbf{v}\} = (i/2)\lambda G(\mathbf{k}, f)P(\mathbf{k}) \int_{-\infty}^{+\infty} df' \int_{k_d e^{-l}}^{k_d} d^D q \mathbf{v}(\mathbf{q}, f') \mathbf{v}(\mathbf{k}-\mathbf{q}, f-f') \tag{7.16}$$

where we omitted all tensor and vector indices since they will play no role in what follows. Only the scaling powers matter.

2. Now we "solve" (7.6) by developing a perturbation expansion for the part of interaction $J^>$ by iterations in powers of λ substituting the velocity field zero order approximation (7.10) into the part of nonlinear term (7.16) to generate $v^{(1)}$ as the first power of λ approximation and then substituting it into the part of nonlinear term (7.16) again to generate the λ^2 approximation and iterating the procedure *ad infinitum*. Therefore the procedure is like in the usual perturbation theory but only a small part of the nonlinear term is treated perturbatively. The remaining part of interaction $J^<$ is remained untouched. The thus generated power series is averaged over the part of

Gaussian force $F^>$ from the shell $k_d \geq q > k_d e^{-l}$. Because the forcing is Gaussian the parts $F^>$ and $F^<$ are statistically independent. This means for instance that $\langle F^> \rangle^{F^>} = F^<$, $\langle v \rangle^{F^>} = v^<$ and the odd powers correlation functions $\langle (F^>)^{2n+1} \rangle = 0$. The result is the following solution for $v^<$ as an iterative expansion in powers of λ :

$$\begin{aligned}
 v^<(k < k_d e^{-l}, f) &= GF^<(k < k_d e^{-l}, f) + \\
 + (-i \frac{\lambda}{2} G(\mathbf{k}, f) P(\mathbf{k}) \int_{>}^l < \mathbf{v}(\mathbf{q}, f') \mathbf{v}(\mathbf{k}-\mathbf{q}, f-f') >^{F^>} + \\
 + \sum_{n=1}^{\infty} (-i \frac{\lambda}{2})^{n+1} v^{(n)} + Q^<\{v^<\}, & \quad (7.17)
 \end{aligned}$$

where:

$$\begin{aligned}
 v^{(n)} &= \int_{>}^n \{P(\mathbf{k}) \dots P(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)})\} \times \\
 &\times \{G(\mathbf{k}, f) \dots G(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)}, f - \sum_{i=1}^{n-1} f^{(i)})\} \times \\
 &< v(\mathbf{q}, f^{(1)}) \dots v(\mathbf{q}^{(n)}, f^{(n)}) v(\mathbf{k}-\mathbf{q}, f - f^{(1)}) \dots \\
 &\dots v(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)}, f - \sum_{i=1}^{n-1} f^{(i)}) >^{F^>} = \\
 &= \int_{>}^n \Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)}) \delta(\sum_{i=1}^{n-1} \mathbf{q}^{(i)}) \delta(\sum_{i=1}^{n-1} f^{(i)}) \times \\
 &\times \{P(\mathbf{k}) \dots P(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)})\} \times \\
 &\{G(\mathbf{k}, f) \dots G(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)}, f - \sum_{i=1}^{n-1} f^{(i)})\} v^<(\mathbf{k}, f), & \quad (7.18)
 \end{aligned}$$

where the appearance of δ - functions is the usual consequence of statistical homogeneity and stationarity of the velocity field and subsequent translational invariance for the correlation functions:

$$\Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)}) \delta(\sum_{i=1}^{n-1} \mathbf{q}^{(i)}) \delta(\sum_{i=1}^{n-1} f^{(i)}) =$$

$$\begin{aligned}
 &= \langle \mathbf{v}(\mathbf{q}, f) \dots \mathbf{v}(\mathbf{q}^{(n)}, f^{(n)}) \rangle^{F^>} ; \\
 &\int_{>}^n = \int \prod_{i=1}^n dq^{(n)} df^{(n)}, \tag{7.19}
 \end{aligned}$$

where $k_d > q^{(n)} > k_d e^{-l}$ and in all orders in(7.19) one substitutes the zero order approximation:

$$v^{<}(k < k_d e^{-l}, f) = GF^{<}(k < k_d e^{-l}, f). \tag{7.20}$$

Therefore $\Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)})$ is factorized as a sum of all permutations of products of $\Phi^{>}$. The remaining term $Q^{<}\{v^{<}\}$ is also an infinite expansion containing all order nonlinearities in powers of $v^{<}(\mathbf{k}, f)$. One can ask a question what is the use of the incredibly complicated looking equation (7.17)? But this is the whole power of RNG. Its prime objective is to establish what happens with the coupling under the iterative application of the RNG steps. If the coupling grows then the problem is not renormalizable. But if does not grow and on the contrary reaches some fixed point and this is a stable fixed point then all the terms in the infinite and seemingly incomprehensible expansion (7.17) have the same extent of singularity that does not increase for the high order terms in powers of coupling constant λ . Usually therefore it is enough to consider the terms only to powers λ^2 .

To simplify matters the perturbation expansion should be considered now in the limit $(k, f) \rightarrow 0$. Considering with attention the product $\{P(\mathbf{k}) \dots P(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)})\}$ and taking into account that $\delta(\sum_{i=1}^{n-1} \mathbf{q}^{(i)}) = 0$, it is easy to notice that in this limit the leading term in powers of $k/q^{(i)}$, where $q^{(i)} \gg k$ is from the shell, is $\propto k^2 P(\mathbf{q}^{(2)}) \dots P(\sum_{i=1}^{n-2} \mathbf{q}^{(i)})$ to all orders of λ . Similarly the product of Green functions in (7.18) simplifies in such a way that only the first and the last of them in the whole product are k dependent, to leading order in powers of $(k/q^{(i)})$ and $f/f^{(i)}$ and to all orders in powers of λ in the limit of $(k, f) \rightarrow 0$. As a result the k dependence in the expansion (7.18) will be the same to all orders of λ . Then it can be readily seen that to all orders the perturbation expansion the leading terms in powers of $(k/q^{(i)})$ and $f/f^{(i)}$ can be represented through renormalized Green function with the self-energy (real):

$$\sum(\mathbf{k}, f) \propto \lambda^2 k^2. \tag{7.21}$$

Effectively the molecular viscosity becomes $\nu \rightarrow \nu(1+O(\lambda^2, l))$. This is fairly remarkable because the perturbation expansion carried out in the asymptotic RNG spirit captured immediately the fact that the nonlinear term naturally renormalizes viscosity. But this is the same as renormalization of the coupling constant $\lambda = \lambda_0 A^{1/2}/\nu^{3/2} = Re$. In other words $Re \rightarrow Re^{ren}(l) < Re$. Since the parameter of renormalization l is infinitesimal this reduction of Re is small. But after many iterations of the shell perturbation scheme the reduction factor may become large in the limit $k \rightarrow 0$ as some singular power law function of k . Note that the forcing does not renormalize to the leading order of the perturbation expansion at all, so that $y \rightarrow y$.

3. To see how it happens one carries out the next step of RNG with the following rescaling $k' \rightarrow ke^l$. The meaning of this step is to return to the original span of wavenumber space $k_d > k' > 0$ and to see if the dynamical equation remains invariant and if yes what then happens with λ . It is now necessary to invoke the time or frequencies scaling like in (7.1). We write for the frequencies exponent:

$$[f] = -z, \tag{7.22}$$

meaning that when $k' \rightarrow ke^l$ at the same time $f \rightarrow fe^{zl}$. Now to avoid confusion in superscripts it is convenient to redefine again $k' = k$ and $f' = f$, but remembering that both are the rescaled values. The rescaling now should be carried out in (7.17) and it is enough for now to do it with the term (7.16) and the left side of (7.17). While the rescaling $(k, f) \rightarrow (ke^l, f \rightarrow fe^{zl})$ the velocity field inside the shell is rescaled via its zero approximation (7.20). Hence:

$$[v^>(\mathbf{k}, f)] = [F^>] + [G] \tag{7.23}$$

$$[G] = [f] = z$$

$$[F^>] = [\Phi]/2 + D/2 + z/2 = (y + D)/2 + z/2,$$

where for the derivation of the relations (7.23) we have used (7.3), (7.9), (7.20), the dimensional properties of δ - functions, $[\delta(\mathbf{k}+\mathbf{k}')] = D$, $[\delta(f+f')] = z$ and the assumption $z < 2$. The coefficient in front of the left side of rescaled (7.17) should be kept equal to unity so that after the rescaling the equations remain invariant. Finally one finds that the nonlinear coupling terms rescale as follows:

$$\lambda \rightarrow \lambda(l) = \lambda \exp\{3/2z - 1 + (y - D)/2\}l. \tag{7.24}$$

It is concluded that $\lambda(l) = Re^{ren}(l)$ remains constant and does not grow with l for the following value of z :

$$z = 2/3 - (y - D)/2. \tag{7.25}$$

In particular for $y = D$ one has $z = 2/3$. This value corresponds to the K41 spectrum as was discussed a number of times before since it corresponds to the eddy viscosity defined by (3.34). But this can be also seen directly from the definition:

$$\begin{aligned} E(k) &= k^2 \int df E(\mathbf{k}, f) = k^{D-1} \int df \Phi^{ren}(\mathbf{k}, f) |G^{ren}(k, f)|^2 = \\ &= k^{D-1} \int df \Phi(\mathbf{k}, f) |G^{ren}(\mathbf{k}, f)|^2. \end{aligned} \tag{7.26}$$

Since $[\Phi^{ren}] = [\Phi] = y$ it follows:

$$[E(k)] = -(-D + 1 + y + z). \tag{7.27}$$

To check the correctness of the above analysis let us see what happens with the bare viscous term under the above RNG manipulations. The $m \rightarrow$ iterations of perturbation expansion and rescaling, the two basic steps of RNG, results in the following for $y = D$:

$$\nu k^2 \rightarrow \nu e^{(z-2)l} k^2 \dots \rightarrow e^{(z-2)ml} \rightarrow \lim_{m \rightarrow \infty} k^{-4/3} k^2 = k^{2/3}. \tag{7.28}$$

The last step that should be done is to check if the fixed point value (7.25) is stable as a function of l , i.e., it is necessary to make sure that $\lambda(l)$ has a minimum as a function of l at the fixed point. To do this one considers the next order terms expansions that are not zero for this fixed value of z . This results in a nonlinear recursion relation for the coupling parameter that shows that it reaches a fixed point if the relation (7.25) is fulfilled. In fact the fixed point for the coupling constant is stable and the subsequent results are true to all orders of the RNG perturbation expansion provided that $y \leq D$ (DeDominicis and Martin, 1979; McComb, 1991).⁵¹

⁵¹Note that the parameter y and D always enter in the combination $\pm(y - D)$. For one used to RNG treatment of static systems, e.g., phase transitions, this is unusual. But the actual "dimensionality" of scaling dynamical systems is this combination. The higher order nonlinearities in particular are all proportional to $(y - D)$ and this is why $y = D$ is a borderline for the stability of RNG procedure. In the next section where we will introduce functional integral representation this point will become quite apparent.

The conclusion is that the infrared limit RNG generates a nontrivial solution corresponding to a seemingly stable fixed point coupling to all orders of the perturbation theory provided that $y \leq D$. However even with this restriction on y there is no guarantee that there is no crossover to another solution in the opposite limit of $(k, f) \rightarrow \infty$, as was pointed out in DeDominicis and Martin (1979). For the border line $y = D$ the spectrum becomes indeed K41. But actually it is the limit of $(k, f) \rightarrow \infty$ that should be looked for if we want to capture the properties of turbulence itself, because the energy cascade goes to the large wavenumbers and frequencies $(k, f) \rightarrow \infty$, and it is in this limit the asymptotic solutions should be sought. If they exist in this limit they can depend on powers of the integral scale $L = k_0^{-1}$, and on the contrary the solutions cannot be dependent on the high wavenumbers cutoff k_d , or equivalently Re , except from possible logarithmic pre-factors. In the next section the ultraviolet RNG analysis will be conducted in detail and shown to furnish very useful results when considered in conjunction with certain basic assumptions regarding the dynamical role of **BCC**.

At this point the most disappointing in the above analysis of RSF model is that it does not really reflect the reality of turbulence. Indeed, it is barren of intermittency and coherence. Any order velocity and velocity derivatives correlation function would have the exponents as must have been in accordance with K41 theory and with intermittency parameters $\mu(n) = 0$. The RNG theory is correct to all orders of perturbation expansion and self-consistent as far as the RSF model is concerned and still irrelevant for turbulence. It picks up a wrong solution. This should have been clear from the start. Indeed, the asymptotic limit that was considered is the infrared limit in the terminology of field theories, $(k, f) \rightarrow 0$. There cannot be a cutoff dependence in this limit. But intermittency and scaling together are not possible unless the "anomalous" powers of k are not matched by the corresponding powers of the integral scale $L = k_0^{-1}$.⁵²

Let us go back to the usual perturbation theory methods that also yielded K41 energy spectrum in their many modifications years before the RNG methods were introduced for the treatment of singular dynamical systems. The advantage of renormalized perturbation theory over RNG is in that it is directly applied to turbulence model

⁵²Except from possible logarithmic pre-factors to power laws.

with forcing (7.5) and not necessarily to RSF model. If the chain of equations (7.13) is broken to orders $(\lambda^{ren})^2$ terms and then the quasi-Gaussian assumption is made one obtains a closed system of equations for the velocity two point correlation function $\langle \mathbf{v}(\mathbf{r}, t)\mathbf{v}(0, 0) \rangle$, and the energy spectrum $E(k)$ by the definition (3.13), Φ^{ren}, G^{ren} and λ^{ren} . This approximation was first analyzed by Kraichnan (1973 and references therein) and called Direct Interaction Approximation (DIA). The meaning of DIA is that among all the interactions between the velocity harmonics the only ones that are important are for the velocity Fourier harmonics with the triads of wave vectors that are of the same order by absolute value. This assumption was already mentioned above in Section If we look at the Navier-Stokes equations (7.6) this would mean the following triads:

$$|\mathbf{k}| \sim |\mathbf{q}| \sim |\mathbf{k}-\mathbf{q}|. \quad (7.29)$$

In fact this approximation seems to be the opposite of RNG analysis previously considered where the interactions between velocity harmonics of all scales were taken into account. But the reasoning behind DIA has sense and this is why. It was long understood that the nature of interactions between turbulent eddies is twofold. The small eddies are embedded inside large eddies like in the Russian dolls. It is clear therefore that the strongest interaction between such disparate sizes eddies is just convection by a large eddy of the small ones. Of course the small eddies are also strained by them. But it seems that there should be no significant flux of energy from the eddies of strongly disparate sizes and in this sense these interactions are not dynamically important. The real exchange of energy should be among the eddies of comparable size and therefore the dynamically significant interactions are primarily local in Fourier space like in (7.29).

In the framework of perturbation theory the justification comes mathematically from the following deep observation. The largest terms in the perturbation expansions (7.13) comes from the terms corresponding on the contrary to the interactions between the velocity harmonics of strongly disparate scales $|\mathbf{k}| \gg |\mathbf{q}|$. These terms appear as expansions in ever increasing powers of $L^{1/3}$ and are unrenormalizable. However Kraichnan suggested and partially justified mathematically that the leading terms should be just an expansion of

the Galilean velocity shift that comes from convection small eddies by the large ones and hence can be removed by Galilean transformation. If to think about it this would be as if one abandons for a time the Eulerian fixed observer perspective of fluid motion in favor of the Lagrangian reference frame co-moving with fluid elements, the small eddies in this case. The difficulty is that the large eddies move randomly and this is why Kraichnan introduced a concept of random Galilean transformation that was supposed to remove the leading singularities in the perturbation expansion. Another difficulty is of course that there are an infinite number of other singularities in the perturbation expansions equally unrenormalizable. It takes a stretch of imagination that all of them can be removed by random Galilean transformation. Such claims were made by some but I disregard them since no doubt in my mind that they contained some trivial mathematical mistakes inevitably made when one tries to make exact statements as regards unrenormalizable perturbation expansions. Nevertheless, the energy transfer is most likely indeed dominated by local interactions in Fourier space at least in the limit of $Re \rightarrow \infty$. The DNS start giving support to the locality principle although it appears that the contribution of distant in Fourier space interactions decreases very slowly with the growth of Re (Mininni, et.al., 2008b).

Nevertheless, if the assumption of wavenumbers space locality is made this inevitably leads to the K41 spectrum. No other result is possible since the terms with L dependence are all removed and then the dimensional considerations and scaling result in K41 unambiguously. It should be noted that although the infrared RNG perturbation theory seemingly take care of distant in k - space interactions this is an illusion. Somewhere along the line such interactions cancel out and solely the interactions of the type (7.29) remain.

Despite its obvious shortcomings DIA has nice properties. It allows calculating the pre-factor Kolmogorov constant that is not far from the experimental. And the Kolmogorov constant allows estimating the von Karman constant for the boundary layers and pipes logarithmic flow profile (6.23). The same even better can be done with the help of infrared RNG theory for RSF model. Clearly DIA and RNG belong on a certain fundamental level to the same class of theories. The fact is that a good number of useful constants characteristic of turbulent flows can be estimated from the drastically

truncated models that all have K41 as their solution.^p

8 Dynamical Theory 2: Perturbation Theory and RNG Analysis in Asymptotic Limit of High Wavenumbers and Frequencies

In this section a totally different approach to the dynamics of turbulence will be considered. It will be based on a number of assumptions that will become clearer as it goes. The central assumption is that turbulence dynamics is dominated by a fractal sub-domain and all 3D turbulent flow is formed and sustained by this sub-domain. It is an indication well supported by DNS and some experimental facts that were discussed above that this sub-domain is **BCC**, but the properties of **BCC** will not play a direct role in the dynamical phenomenology developed below. The main objective here will be to determine the fractal dimension of the sub-domain while its existence is presupposed.

It is well known that the turbulence model described by the Navier-Stokes equations with Gaussian random source (forcing) can be reformulated in the language of functional integrals (e.g., Monin and Yaglom, 1975). Admittedly functional integrals have not been very useful so far for real calculations. But they provide sometimes a different way of interpreting phenomena and this may be conducive for making and interpreting specific assumptions that will be made below. A particular way to formulate the problem for turbulence was developed in Levich (1987) and this paper is referred to for some technical details.

The RSF model is the simplest to reformulate as a functional integral probabilistic problem of the type (2.2). Following the general definition (2.2) let us consider first the following functional:

$$Z = \int W\{\mathbf{v}\} D\mathbf{v} = \lim_{\Delta \rightarrow 0} N(\Delta)^{-1} \int D\mathbf{v} \exp\{-\Theta^{(2)}\{\mathbf{v}/\Delta\}\}, \quad (8.1)$$

where $\Theta^{(2)}$ is the functional:

$$\Theta^{(2)} = \int dV dt \{ \partial_t \mathbf{v} - \mathbf{J}\{\mathbf{v}\} - \nu \Delta \mathbf{v} + \mathbf{F} \}^2;$$

$$\mathbf{J}\{\mathbf{v}\} = [\mathbf{v} \times \boldsymbol{\omega}] - \nabla(P + \mathbf{v}^2/2), \quad (8.2)$$

Where the integration $D\mathbf{v}(\mathbf{r}, t)$ in (8.1) is, as in (2.2), over all $\mathbf{v}(\mathbf{r}, t)$ turbulent flow realizations, each realization being the velocity field in all (3+1) space/time. \mathbf{F} is the external source at large scales defined as in (7.5) that is assumed to be a Gaussian random function defined as in corresponding to Gaussian forcing at large scales defined through the correlation function as in (7.5), $N(\Delta)^{-1}$ is a normalizing factor and $\Delta \rightarrow 0$ is a parameter that can be likened to temperature in equilibrium systems. It is easy to prove that the integrand in (8.1) is a functional $\delta - function$. Indeed, let us chose a new integration variable:

$$\mathbf{X}\{\mathbf{v}\} = \partial_t \mathbf{v} - \mathbf{J}\{\mathbf{v}\} - \nu \Delta \mathbf{v}. \quad (8.3)$$

Then for Z one has:

$$\begin{aligned} Z &= \int W\{\mathbf{v}\} D\mathbf{v} = \\ &= \lim_{\Delta \rightarrow 0} N(\Delta)^{-1} \int D\mathbf{X} \det\{\mathbf{v}(\mathbf{r}, t)/\mathbf{X}\} \exp\{-\Theta^{(2)}\{\mathbf{v}/\Delta^2\}\}. \end{aligned} \quad (8.4)$$

But the *Det* by a change of functional variables can be always made unity. To prove it is necessary to consider the discretized velocity field model on a lattice. The choice of lattice grid can be arranged in such a way that $\det\{\mathbf{v}(\mathbf{r}, t)/\mathbf{X}\}$ is independent of $\mathbf{v}(\mathbf{r}, t)$ and can be absorbed into redefined normalizing pre-factor. Thus the integrand in Z is indeed a functional $\delta - function$:

$$\begin{aligned} Z &= \lim_{\Delta \rightarrow 0} N(\Delta)^{-1} \int D\mathbf{v} \exp\{-\Theta^{(2)}\{\mathbf{v}/\Delta^2\}\} = \\ &= \int D\mathbf{v} \delta\{\mathbf{v} - \mathbf{v}_{NS}\} = 1. \end{aligned} \quad (8.5)$$

Hence as far as the statistical properties (2.3) are concerned the Eq. (8.1) is an identical reformulation of the Navier-Stokes equations for RSF model. The only contributions into Z will be the velocity realizations that are the solutions of the Navier-Stokes equation $\mathbf{v}_{NS}\{\mathbf{F}\}$.

The $Z - functional$ can be reformulated for the velocity field in Fourier space. One has evidently:

$$\Theta^{(2)} = \int (\mathbf{X})^2 dV dt = (2\pi)^{-3} \int |\tilde{\mathbf{X}}(\mathbf{k}, f)|^2 d\mathbf{k} df, \quad (8.6)$$

where:

$$X_i(\mathbf{k}, f) = (-if + \nu k^2)v_i(\mathbf{k}, f) + (i\lambda/2)P_{ijs}(\mathbf{k} \int d^D q d f' v_j(\mathbf{k}, f)v_s(\mathbf{k}-\mathbf{q}, f-f') + F_i(\mathbf{k}, f). \quad (8.7)$$

Therefore:

$$Z = \lim(\Delta \rightarrow 0)N'(\Delta)^{-1} \int D\mathbf{v}(\mathbf{k}, f) \exp\{-\Theta^{(2)}\{\mathbf{v}\}/\Delta^2\}, \quad (8.8)$$

where $N'(\Delta)^{-1}$ is another normalizing pre-factor. It is natural to average Z over all realizations of the Gaussian external force, since the velocity correlation functions in turbulence are independent of the force realization. In RSF model this is not true as we have seen but in this model the dependence is only on the scaling properties of $\Phi(k, f) = A/k^\gamma$ as given above by the relation (7.9). We define $\langle Z \rangle^F$ as follows:

$$\langle Z \rangle^F = \int DFW\{F_i\} Z = \int DF \exp\{-\int d^D k d f \Phi_{ij}^{-1}(k, f) F_i F_j\} Z \quad (8.9)$$

where we substituted for $W\{F - i\}$ the Gaussian probability functional and the pre-factor Φ_{ij}^{-1} is determined by (7.3) and can be interpreted as "temperature":

$$\Phi_{ij}^{-1}(\mathbf{k}, f) = (\delta_{ij} - k_i k_j / k^2) \Phi^{-1}(k, f). \quad (8.10)$$

The integration of the Gaussian functional is trivial with the result (hereafter the normalizing pre-factors are neglected):

$$\langle Z \rangle^F = \int d\mathbf{v}(\mathbf{k}, f) W(\mathbf{v}, \Phi), \quad (8.11)$$

With the probability distribution:

$$W(\mathbf{v}, \Phi) = \exp(-\Theta_\Phi^{(2)}), \quad (8.12)$$

where:

$$\Theta_\Phi^{(2)} = \int d^D d f \mathbf{X}'(\mathbf{k}, f) \mathbf{X}'(-\mathbf{k}, -f) / \Phi(k, f), \quad (8.13)$$

where:

$$X'_i(\mathbf{k}, f) = (-if + \nu k^2)v_i(\mathbf{k}, f) + (i\lambda/2)P_{ijs} \int d^D d f' v_j(\mathbf{k}, f)v_s(\mathbf{k}-\mathbf{q}, f-f'). \quad (8.14)$$

And finally since the concern is the velocity (and other turbulent fields) correlation functions we can write formally like in (2.3) but now with the specific probability distribution functional $W\{\mathbf{v}, \Phi\}$.⁵³

$$\langle \prod_{i=1}^n (\mathbf{r}_i, t_i) \rangle = \int D\mathbf{v} \prod_{i=1}^n \mathbf{v}(\mathbf{r}_i, t_i) W\{\mathbf{v}, \Phi\}. \quad (8.15)$$

What has been achieved by the above identical transformations? It seems not much, but still we created a finite width in the probability distribution function (8.12). Now instead of the previously zero width $\Delta \rightarrow 0$ we have "temperature" that is a scaling function in (k, f) - space. The functionals $X'_i(\mathbf{k}, f)$ and $\Theta_{\Phi}^{(2)}$ are now independent of the external force $F_i(\mathbf{k}, f)$ and since the "temperature" in (8.13) is not zero everywhere in $(, d)$ space it is evident that the contribution to $\langle Z \rangle^F$ comes not necessarily only from the decaying with time solutions of the unforced Navier-Stokes equations $X'_i(\mathbf{k}, f) = 0$. Since a functional integral unless it is a Gaussian one is not possible to calculate it still appears as if the mathematical equilibristics that leads to them remains exactly this-the equilibristics. Of course a perturbation theory can be developed for the form (8.15) but it would be equivalent to the perturbation theory analysis of certain differential equations. Since the functional integrals are good only for the calculation of correlation functions it may seem that much less information is required about the velocity field and this would result in simplifications for instance of DNS. In practice however it does not happen and everyone still prefers to carry out the DNS of the Navier-Stokes equations themselves. One of the main reasons why the functional integral forms are used in this paper is that interpretations that they allow are sometimes easier to understand than the ones furnished by

⁵³Or the correlation functions can be expressed as functional derivatives of $\langle Z \rangle^F$. In the same way can be defined all other velocity and velocity derivatives correlation functions and structure functions. Since no particular calculation will be made using the functional integral language we skip these particular definitions as not used.

local in space differential equations. But in Section 9 below we will show how the functional integral representation allows a global approach and new results that are difficult to get within the local in space/time differential representation.

Let us try first to interpret the RSF models and their relation to real turbulence model from the perspective of functional integrals. It seems that since for turbulence model the effective temperature $\Phi(k > k_0, f) \rightarrow 0$ the only and very few realizations of $\mathbf{v}(\mathbf{k}, f)$, almost purely decaying "low temperature" solutions of the Navier-Stokes equations, $|\mathbf{X}|^2 k^3 f < \Phi(k > k_0, f) \rightarrow 0$, would contribute to the correlation functions (8.15), since the integrand in the functional integral tends to a functional δ - *function*. But this is not exactly true. Since in general any realization of $\mathbf{v}(\mathbf{k}, f)$ should be taken into account for calculating (8.15) there is an infinity of realizations that are far from the decaying solution of the Navier-Stokes equations that would contribute by their sheer number, although it may be that each particular contribution would be exponentially small. If these "non-classical" realizations are taken into account first, and not the limit $\Phi(k > k_0, f) \rightarrow 0$, then the result in principle can be different. This would imply non-analyticity in the procedure of calculations of the correlation functions and since there is only one parameter in the problem, the Reynolds number, it should be safely assumed that such non-analytical contributions may arise in the limit $Re \rightarrow \infty$. What one can say about $\mathbf{v}(\mathbf{k}, t)$ - *realizations* such that $\Theta_{\Phi}^{(2)} \gg 1$? Formally, such configurations can be assumed solutions of the renormalized Navier-Stokes equations $X'_i(\mathbf{k}, f) = F'_i$, where F'_i is generally unknown random function. This function determines the extent of deviation from the decaying solutions $X'_i(\mathbf{k}, f) = 0$; it can be seen as renormalized driving force in the Navier-Stokes equations. Let us decompose F'_i into a Gaussian and non-Gaussian parts: $\mathbf{F}' = \mathbf{F}'_{reg} + \mathbf{F}'_{sing}$, where we designated *reg* and *sing*, respectively for quasi-Gaussian and non-Gaussian, generally intermittent, velocity field components for a purpose that will become clear shortly. Thinking consistently in the spirit of RNG shell integration we can formulate a new generating functional for the renormalized Navier-Stokes equations with the integrand as follows:

$$W_{ren}(\mathbf{v}, \Phi_{ren}) = \exp(-\Theta_{\Phi, ren}^{(2)}),$$

$$\Theta_{\Phi,ren}^{(2)} = \int d^D df |\mathbf{X}'(\mathbf{k}, f) - \mathbf{F}'_{sing}|^2 / \Phi(k, f)_{ren}, \quad (8.16)$$

where $\Phi(k, f)_{ren}$ is effectively a new renormalized temperature distribution determined by the forcing \mathbf{F}'_{reg} and corresponding to a higher level of fluctuations in the interval $ke^l > k > k_0$. The velocity correlation functions is determined with the new probability distribution (8.16). The total number of $\mathbf{v}(\mathbf{r}, t)$ - *realizations* that would contribute to the generating functional is now less because some of them were already taken into account and resulted in the renormalized probability distribution. The iteration of this reasoning should further diminish the number of these realizations while giving rise to temperature at respectively higher values of the wavenumbers *ad infinitum*. The assumption is that such reasoning results in a fixed point corresponding to such forcing that there would be only finite number of $\mathbf{v}(\mathbf{r}, t)$ - *realizations* left to contribute to the correlations functions. The assumption of scaling tells us that if such fixed point exists then the forcing corresponding to this fixed point should be the scaling function (7.9).

Let us discuss now the meaning of \mathbf{F}'_{reg} and \mathbf{F}'_{sing} . It starts from the splitting of the velocity field in physical space in every realization into two components:

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t)_{reg} + \mathbf{v}(\mathbf{r}, t)_{sing}, \quad (8.17)$$

where it is assumed that $\mathbf{v}(\mathbf{r}, t)_{sing}$ is the intermittent velocity field inside small sub-domains that in the limit $Re \rightarrow \infty$ will tend to a fractal space with dimension $D < 3$ and $\mathbf{v}(\mathbf{r}, t)_{reg}$ is the velocity field outside of these small sub-domains. We assume further that the velocity field $\mathbf{v}(\mathbf{r}, t)_{reg}$ is quasi-Gaussian in the spirit of all phenomenological fractal theories of the Section 5.⁵⁴ And here we make the central assumption that the total contribution of all $\mathbf{v}(\mathbf{r}, t)_{sing}$ into the correlation functions (8.15) is in a certain sense of the same order as of all $\mathbf{v}(\mathbf{r}, t)_{reg}$ - *sub-relations*. Let us try to define this

⁵⁴Attention should be paid to the fact that old order correlation functions are not necessarily zero in the quasi-Gaussian approximation. If, for instance, a certain third order velocity correlation function in Fourier space would be zero the nonlinear transfer of energy would be zero. But what is true in the quasi-Gaussian approximation is that any odd order correlation function is expressed eventually via the pair correlation functions so that a closure can be implemented.

assumption in a quantitatively useful way. Let us rewrite (8.15) as follows:

$$\langle \prod_{i=1}^n \mathbf{v}(\mathbf{r}_i, t_i) \rangle = \int D\mathbf{v}_{sing} D\mathbf{v}_{reg} \prod_{i=1}^n \mathbf{v}(\mathbf{r}_i, t_i) W\{\mathbf{v}, \Phi_{reg}\}. \quad (8.18)$$

Instead of calculating the functional integral over all possible $\mathbf{v}(\mathbf{k}, t)_{sing}$ - *sub-realizations* we shall try to estimate the contribution of a certain averaged singular sub-realization in a way reminiscent of *mean field* approximation in statistical mechanics. This approximation mathematically looks as follows:

$$\int D\mathbf{v}_{sing} W\{\mathbf{v}_{reg}, \mathbf{v}_{sing}, \Phi_{reg}\} \approx \exp[-\langle\langle \Theta^{(2)} \rangle\rangle], \quad (8.19)$$

where the averaging $\langle\langle \dots \rangle\rangle$ is done over all $\mathbf{v}(\mathbf{k}, t)_{sing}$ - *sub-realization*. In consequence for the correlation functions it will be now:

$$\langle \prod_{i=1}^n \mathbf{v}(\mathbf{r}_i, t_i) \rangle = \int D\mathbf{v}_{reg} \prod_{i=1}^n \mathbf{v}(\mathbf{r}_i, t_i)_{reg} W\{\mathbf{v}_{reg}, \Phi_{reg}\}, \quad (8.20)$$

where the reduced probability distribution functional is as follows:

$$W\{\mathbf{v}_{reg}, \Phi_{reg}\} = \exp[-\langle\langle \Theta_{\Phi_{reg}}^{(2)} \rangle\rangle] \quad (8.21)$$

and:

$$\langle\langle \Theta_{\Phi_{reg}}^{(2)} \rangle\rangle = \int d^D df \Phi^{-1}\{|\mathbf{X}_{reg}|^2 + \langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle\}. \quad (8.22)$$

The \mathbf{X}_{reg} part is obtained by the substitution $\mathbf{v} \rightarrow \mathbf{v}_{reg}$ into \mathbf{X}' . Assume that we develop a perturbation theory for calculating of the correlation functions (8.20). To do this in the language of functional integrals we must expand the integrand in (8.20) in powers of the coupling parameter in front of the coupling terms in $\langle\langle \Theta_{\Phi_{reg}}^{(2)} \rangle\rangle$. In this expansion all the powers of coupling terms from both $|\mathbf{X}_{reg}|^2$ and $\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle$ would be present. Assume that somehow this expansion is renormalizable. If scaling is as usual also assumed it means that effectively all the most singular terms in the limit $(k, f) \rightarrow \infty$ in the expansion would have the same scaling exponents that would not

increase with the increase of n . In the above sense the functionals $|\mathbf{X}_{reg}|^2$ and $\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle$ themselves can be considered as scale invariant functionals. In other words when most contributing velocity field realizations are considered the scaling transformations (7.1) should generate an appropriate scaling transformation of $|\mathbf{X}_{reg}|^2$ and $\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle$, i.e.:

$$[|\mathbf{X}_{reg}|^2] = e_{reg}(z)$$

and

$$[\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle] = e_{sing}(z) \tag{8.23}$$

The assumption of equal contribution will be now simply:

$$e_{reg} = e_{sing}, \tag{8.24}$$

or again in the language of a hypothetical at this point renormalizable perturbation expansion the most singular terms in the limit $(k, f) \rightarrow \infty$ that would be generated by the coupling from $\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle$ would be not more singular in this limit than the ones generated by the coupling in $|\mathbf{X}_{reg}|^2$.

In fact this scaling equality seems inevitable. Because if one or another components, regular or singular, has a larger scaling exponent then by iterative application of scaling the role of the remaining component will be scaled out together with its dynamical contribution. In other words either the regular part of the velocity field would be a dominant contribution to the velocity correlation functions of all orders or on the contrary the singular fractal component will dominate in all orders of the correlation functions. Neither option is satisfactory. The first one would mean that there is no intermittency in all orders. The second one would mean that the fractal component is actually the only one that matters even for the pair correlations function (and for the K41 spectrum in consequence). But we know that the velocity field is nearly Gaussian as far as the second order two point correlation function or equivalently the second order structure function are concerned. The equality of contributions in the scaling sense is the only viable option. The regular and singular components of turbulence must be in a certain dynamical equilibrium with each other. The quasi-Gaussian component $\mathbf{v}(\mathbf{r}, t)_{reg}$ itself should be rather seen as generated by the fractal component $\mathbf{v}(\mathbf{r}, t)_{sing}$. Such

in fact is the only logical interpretation of (8.17). Indeed, $\mathbf{v}(\mathbf{r}, t)_{sing}$ has by definition the most of vorticity, vorticity generation and other velocity derivatives dependent quantities that are in Fourier space are defining the high wavenumbers activity. And by the analogue of Bio-Savart induction theorem this vorticity field creates a flow field around it in the whole domain (see the discussion in Section 1 and footnote 39 in Section 5).

As far as the singular part dependent contributions it will be assumed that sub-ensemble $\overline{\cdot}$ averaging is equivalent to the space averaging, i.e., effectively to the smearing of the fractal sub-domain over the whole $3D$ space. Hence for instance we assume:

$$\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle = \lim_{V \rightarrow \infty} V^{-1} \int dV_F |\mathbf{X}_{sing}|^2, \quad (8.25)$$

where $dV_F = d^{D_F} \mathbf{r}$ in (8.25) is a differential of a fractal "volume" as was introduced in (5.6) and (5.8), and similarly for all other $\mathbf{v}(\mathbf{r}, t)_{sing}$ dependent quantities. What is important (and obvious) that the averaging (8.25) results in smearing of the dynamical action $\mathbf{v}(\mathbf{r}, t)_{sing}$ over the total flow domain that results in a scaling factor that tends to zero in the limit of high wavenumbers. This is the best to understand using a discrete description. The vanishing volume sub-domain that we associate with a fractal in a discrete description of Section 4 corresponds to a vanishingly small, by comparison with the total, number of points on a lattice. In Fourier space this would correspond also to a reduction of the number of points on the inverse lattice: the clusters scale on the inverse lattice would grow in correspondence with reduction of clusters scale in physical space. If for $3D$ inverse lattice, as well as the physical space lattice the number of grid points is $\propto k_d^D$, the fractal with physical space dimension D_F is defined by a reduced number of points $\propto k_d^{D_F} = k_d^{D-\mu}$. This scaling reduction is very important in the first place as it tells us that the actual number of degrees of freedom of the velocity field is less than would be for the total phase volume k_d^3 (see the discussion in Sections 3 and 4). In terms of $\mathbf{v}(\mathbf{r}, t)_{reg}$ it means effectively the reduction of the non-linear coupling λ^2 in the functional (8.22). It must be reminded that it is hierarchical Beltramization of the flow that is the genuine reason for the nonlinear coupling reduction and the subsequent formation of the **BCC** fractal itself. However in the mean field approximation

this does not play a direct role.

It was mentioned before and should be pointed out again that it follows from the observations of geophysical turbulent flows with genuinely high Reynolds numbers that the probability distributions for various quantities dominated in Fourier space by high wavenumbers, such as the velocity field derivatives and vorticity in particular, have power law algebraic tails for high deviations from the mean values. It means for instance that the high order velocity field structure functions (5.3) and velocity field derivatives correlation functions do not have physical meaning. They are divergent in the limit of $Re \rightarrow \infty$. Of course for all finite values of Re they do exist and finite, but starting from some order they lose physical meaning and just reflect the algebraic properties of the tails of the probability distribution functions for large excursions of the fields from their mean values. As explained in the works of Lovejoy and Schertzer (e.g., Schertzer and Lovejoy, 1983; Schertzer and Lovejoy, 1985a and Schertzer and Lovejoy, 1985b; see also Levich, 1987 for simplified analysis) the indication of this happening would be the linear growth of the anomalous scaling exponents with the order n of the structure functions, i.e. $\mu(n) \propto n$.⁵⁵ Although the algebraic tails for the probability distribution functions can be given statistical interpretation it is my viewpoint that this merely reflects the highly coherent internal structure of *BCC*. It seems quite out of place trying to describe such superbly ordered flows using statistical language of high order correlation functions. But as soon as the mean field approximation was implemented the issue of high order statistics is not relevant anymore, because the functional (8.22) depends now only on the regular component of the velocity field. And in consequence the probability distribution functional (8.21) is good only for determination of

⁵⁵This linear growth is spurious and reflects the non-existence of physically meaningful high order moments. It should not be confused with the linear growth of intermittency exponents of FHT model (5.11). Turbulence is not adequately described by FHT model. In a paper of Sreenivasan and Antonia (1997) it is argued that the algebraic probability distribution functions for turbulent fields, or as they are also called hyperbolic distributions, are in contradiction with experiment. It is not in my view a proven argument. The true experiment should be carried out for the Re by far larger than possible in laboratory conditions. On the other hand the geophysical data is clearly compatible with the algebraic tails. To attribute this fact to some special properties of geophysical turbulence would be hard to accept.

the structure functions of $\mathbf{v}(\mathbf{r}, t)_{reg}$ and the correlation functions of the derivatives of $\mathbf{v}(\mathbf{r}, t)_{reg}$. But these are all trivial in a way since $\mathbf{v}(\mathbf{r}, t)_{reg}$ is by definition quasi-Gaussian. Therefore for instance the second order structure function (5.3) for $n = 2$ and the corresponding to it energy spectrum in Fourier space should be K41; no other result can be true for the quasi-Gaussian approximation as was explained in the previous Section 6.⁵⁶ And the higher order structure and correlation functions should be all just as prescribed by the original K41 theory with the intermittency parameters all equal to zero. Clearly there is no sense in the above mean field approximation as far as the calculation of the real velocity field high order statistics is concerned because the latter is dominated by $\mathbf{v}(\mathbf{r}, t)_{sing}$ component of the velocity field. As soon as the mean field approximation (8.21) was made the possibility of obtaining correct high order statistical expressions for the velocity field were forfeited and in actual fact the trivial behavior for high order statistics of $\mathbf{v}(\mathbf{r}, t)_{reg}$ no intermittency and no coherence, was imposed.

The question then arises what is the sense of accepting an approximation that cannot furnish anything but the K41 spectrum and similar quasi-Gaussian results for the rest of the correlation functions that were assumed from the beginning in the nature of approximation? The answer to this is that by assuming the K41 spectrum we may be able to calculate $dV_F = d^{D_F} \mathbf{r}$, i.e., the volume of the *v - singular* sub-domains, which is effectively the fractal dimension D_F that is compatible with the K41 spectrum. In this sense the meaning of the K41 spectrum becomes totally different from that implied by K41 theory. Indeed, instead of being the result of homogeneous cascade it becomes a result of the intermittent inhomogeneous nature of turbulence.

Looking ahead we shall determine that in the ultraviolet limit the RNG asymptotic scaling solution of RSF model to be compatible with the K41 spectrum requires a unique renormalization of the coupling parameter λ in the ultraviolet limit $(k, f) \rightarrow \infty$. As was pointed out before this reduction can be interpreted as the smallness of the sub-domain where the $\mathbf{v}(\mathbf{r}, t)_{sing}$ component is located, with the effective

⁵⁶The energy spectrum is the Fourier transform of $\langle \Delta \mathbf{v}^2 \rangle$, as it is the Fourier transform of the two point velocity correlation function (3.13). The two Fourier transform obviously coincide.

volume $dV_F = d^{D-\mu}\mathbf{r}$. In the mean field approximation the action of $\mathbf{v}(\mathbf{r}, t)_{sing}$ is smeared over the whole flow while this flow is itself generated by $\mathbf{v}(\mathbf{r}, t)_{sing}$. From now on there is only $\mathbf{v}(\mathbf{r}, t)_{reg}$ left and it is present in the whole flow domain. The question one asks then is how many independent velocity harmonics, or degrees of freedom it possesses in conjugate Fourier space. Apparently it is the same as the one for the incipient basic $\mathbf{v}(\mathbf{r}, t)_{sing}$ and this is $\propto k_d^{D_F} = k_d^{D-\mu}$, as was discussed previously. And this is exactly the scaling reduction factor $k^{-\mu}$ that we expect to appear in front of the renormalized coupling in the functional (8.22).⁵⁷ Effectively it can be interpreted as a reduction of the square of the book keeping parameter λ^2 (see (7.11) for the definition in RSF) in front of the nonlinear coupling terms in (8.22). Naturally this renormalization must appear explicitly as a result of a proper RNG asymptotic analysis implemented for the functional (8.22). However, we can push further this qualitative reasoning even without doing a proper analysis. Indeed, we know that from the basic scaling properties the effective dimension of the RSF is $(D - y)$. It means that we should have λ^2 renormalized as follows:

$$\lambda^{2ren} \rightarrow \lambda^2 k^{(D-y-\mu)} = \lambda^2 k^{(D_F-y)}. \quad (8.26)$$

We have now two unknown constants to determine, y and μ . But actually it is y and since it is the dynamical exponent z that appears in the scaling analysis. It was defined previously in (7.22) and before since the eddy viscosity concept was first introduced in (3.33). The renormalization of viscosity plays central role in theories (and practice) of turbulence and the eddy viscosity definition and properties should be continuously reminded.

In the framework of K41 theory $z = 2/3$ from a type of dimensional considerations that exclude the possibility of dependence on the integral scale $L = k_0^{-1}$. It was assumed above that the energy spectrum of $\mathbf{v}(\mathbf{r}, t)_{reg}$ is indeed K41 as a concomitant of its quasi-Gaussian nature. Then for RSF model the general relation that connects y, z and the spectrum exponent $[E(k)]$ is (7.27). Choosing $[E(k)] = -5/3$

⁵⁷This is a subtle point because one can expect that the nonlinear coupling generates new harmonics. But the meaning of the whole procedure is that in the accepted approximation the remaining effect of nonlinear coupling is quasi linear and this is a collorary of the scaling assumption (8.24).

and formally defining:

$$z = 2/3 + \mu/3 \quad (8.27)$$

we arrive at the relation:

$$y = D - \mu/3. \quad (8.28)$$

What it shows is that generally the K41 spectrum can be realized for RSF model even if $\mu \neq 0$ and $z \neq 2/3$ if y and z are appropriately related by (8.27) and (8.28). In Section 6 the infrared RNG perturbation analysis and the usual renormalized perturbation theories that all explicitly or implicitly assume that only local in k -space interactions (7.29) play a dynamical role we were driven by the subsequent mathematics to inevitable $\mu = 0$ and $z = 2/3$. But this was a result of by far more stringent assumptions *equivalent* to the assumption of locality of interactions (7.29).⁵⁸ The locality condition precludes any dependence on the infrared cutoff $k_0 = L^{-1}$. But if $z \neq 2/3$ it follows from dimensional considerations that f under a scaling transformation behaves as $k^{2/3+\mu/3}(L)^{\mu/3}$. The K41 spectrum assumption $[E(k)] = -5/3$ does not by itself impose the assumption of locality (7.29) as far as the frequency dependence is concerned. The interaction can be local in k -space but non-local in (k, f) -space.

Now we go back to the scaling relation (7.24). As long as the exponent z is not defined it can be safely used since this is just a scaling property of RSF model, a particular one from the general scale invariant transformations (7.1), independent of the perturbation analysis. Whether we are considering the differential equations or the functional integral representation the scaling relations remain valid and the same. It is only a change of sign for the exponents that is convenient to make since in the following formal analysis we shall be moving with the iterative rescaling into the ultraviolet direction $(k, f) \rightarrow \infty$, instead of the infrared direction $(k, f) \rightarrow 0$. With this comment in mind and substituting for z and y respectively (8.27) and (8.28) we arrive at the following result for the coupling parameter reduction:

$$\lambda^{ren} \rightarrow \lambda k^{-\mu/3}. \quad (8.29)$$

⁵⁸The emphasis on *equivalent* should be noted because infrared RNG in Section 7 formally takes the account of distant interaction in k -space as well. But it is an illusion since eventually their contribution turns out to be exactly the same as if from the start it would be assumed that only local interactions are contributing.

We are left with the determination of one parameter μ . This cannot be done from purely qualitative considerations and require a proper RNG perturbation theory analysis.

At this point we will depart from the formalism of functional integrals since they have no advantages for the specific calculation of μ , although all the calculations that will be made below can be made of course directly in the framework of the functional formulation (8.21).

Let us formulate equations for $\mathbf{v}(\mathbf{r}, t)_{reg}$ such that would lead to the functional integral representation (8.19)-(8.21) in the same way that the Navier-Stokes equations for RSF (7.6) led to the functional integral formulation (8.16). Consider the following model equations:

$$v_{reg,i} = \Xi_{reg,i}G(\mathbf{k}, f) - (i\lambda_0/2)G_{reg}(\mathbf{k}, f) \int d^D q d f \eta_{ijs}(\mathbf{k}, f; \mathbf{q}, f') v_{reg,j}(\mathbf{q}, f') v_{reg,s}(\mathbf{k}-\mathbf{q}, f-f') + \sum_n O(\lambda^{n>1}), \tag{8.30}$$

where:

$$G^{reg}(\mathbf{k}, f) = \{-if + \nu k^2 + \sum(\mathbf{k}, f)\}^{-1} \tag{8.31}$$

and $\eta_{ijs}(\mathbf{k}, f; \mathbf{q}, f')$ is a coupling function that is all unknown to us but its scaling properties. Only the latter will be important in the calculations to follow. Thus we write:

$$\eta_{ijs} = P_{ijs}(1 + \alpha); \tag{8.32}$$

$$[\alpha] = 0; [P_{ijs}] = 1,$$

where P_{ijs} is the usual projection operator (7.8) and the choice of η_{ijs} is made conveniently to satisfy the divergence free condition for $\mathbf{v}(\mathbf{r}, t)_{reg}$, i.e., in Fourier space: $\mathbf{k} \cdot \mathbf{v}(\mathbf{k})_{reg=0}$. But let us continue defining the different terms in the model equations (8.30). It should be pointed out that in fact there is a lot of arbitrariness allowed in (8.30). Except from the scaling relation (8.23) we know nothing about $\ll |\mathbf{X}_{sing}|^2 \gg$. We rely on the assumption that for the determination of certain particular scaling properties nothing else is required. To be consistent we should assume that any (almost)

expression can be added to either the functional $\langle\langle |\mathbf{X}_{sing}|^2 \rangle\rangle$ or the model equations (8.30) as long as the scaling properties are preserved. In this line of thinking we added an eddy viscosity term $\Sigma(\mathbf{k}, f)$ in the Green function (8.31). We assume hence that:

$$[\Sigma(\mathbf{k}, f)] = [f] = [k^z] = z;$$

$$z = 2/3 + \mu/3, \tag{8.33}$$

where z as before in Section 7 is a dynamical scaling exponent that will have to be determined. The value $z = 2/3$ as is usual corresponds to K41.

The $\sum O(\lambda^{n>1})$ terms can be any higher order expansion in powers of the velocity field and it's (nine) derivatives, but such that no term generates higher order nonlinearities in (8.30) than the usual quadratic nonlinear term. It should be considered however that the perturbation expansion of the quadratic nonlinearity itself will generate all higher powers of nonlinearities and in this sense $\sum_n O^{n>1}$ can be safely disregarded from now on. Taking into account all the previous assumptions it is clear now that the functional integral representation (8.19) (8.24) would correspond to the model Eqs. (8.30). As far as the forcing term Ξ_{reg} is concerned then again, since we are only after the scaling exponents, it can be chosen as coinciding with RSF forcing F_i defined by (7.9). Now we will develop the perturbation theory for the Eqs. (8.30) similar to what was done for the RSF model but in the opposite asymptotically ultraviolet limit $(k, f) \rightarrow \infty$.

As in (7.14) let us consider a shell in k -space:

$$k_0 < q < k_0 e^l, \tag{8.34}$$

where instead of starting from the ultraviolet cutoff k_d as in Section 7 we start now from the "infrared" cutoff k_0 and move up in k -space. As before for the infrared RNG limit we split the coupling term in (8.35) into two parts (from now on we omit the subscript *reg*):

$$J_i\{\mathbf{v}\} = J_i^>\{\mathbf{v}\} + J_i^<\{\mathbf{v}\}, \tag{8.36}$$

where now we using some symbolic designations (compare with the definition (7.16) for the infrared asymptotic limit):

$$J_i^<\{\mathbf{v}\} = (i/2)\lambda G^{ren}(\mathbf{k}, f) \times$$

$$\begin{aligned} & \times \int_{-\infty}^{+\infty} df' \int_{k_0}^{k_0 e^l} d^D q \eta(k, f, q, f') v(\mathbf{q}, f') v(\mathbf{k}-\mathbf{q}, f-f') = \\ & = (i/2)\lambda G \int^< \eta v(\mathbf{q}, f) v(\mathbf{k}-\mathbf{q}, f-f'). \end{aligned} \quad (8.37)$$

We shall go on simplifying (8.37). Indeed, since only the scaling properties are sought why to continue with the unknown, except for its scaling exponent, vertex and not deal just with the linear in k vertex $P(\mathbf{k})$ as in the Navier-Stokes equations? By virtue of (8.32) this should not matter as far as the scaling exponents are concerned. And the same should be true for the self-energy part $\Sigma(\mathbf{k}, f; \mathbf{q}, f)$ which we shall substitute by the simplest scaling form:

$$\begin{aligned} \Sigma & \rightarrow k^z = k^{2/3+\mu/3}; \\ z & = 2/3 + \mu/3. \end{aligned} \quad (8.38)$$

We iterate the part of the coupling $J_i^<\{\mathbf{v}\}$ in powers of λ , leaving the remaining part of the coupling term $J_i^>\{\mathbf{v}\}$ untouched. The result of iteration is then averaged over the part of the random forcing $F^<$ from the shell while assuming that $F^>$ is statistically independent of $F^<$, by virtue of being Gaussian, $\langle F^{2n} \rangle = \langle F^2 \rangle^n$, so that $\langle F^> \rangle^{F^<} = F^>$, $\langle F^< \rangle^{F^<} = 0$ and $\langle v \rangle^{F^<} = v^>$; exactly as in Section 6 but in reverse. The iteration and averaging then leads to the following:

$$\begin{aligned} v^>(k > k_0 e^l, f) & = GF^>(k > k_0 e^l, f) + \\ & + (-i\lambda/2)GP \int_l^< v(\mathbf{q}, f^<) v(\mathbf{k}-\mathbf{q}, f-f^<) \rangle^{F^<} + \sum_{i=1}^{\infty} (-i\lambda/2)^{n+1} v^{(n)} + \\ & + Q^>\{v^>\}, \end{aligned} \quad (8.39)$$

where we omitted a number of superscripts and subscripts for the sake of simplicity as they are not important at all for the determination of the scaling properties that we are after. As with the infrared treatment of the previous section the iteration generates all higher order nonlinearities in powers of $v^>$ in the equation (8.39) that are denoted now as $Q^>\{v^>\}$. But the higher order nonlinearities can be shown to be all irrelevant in the RNG analysis which becomes clear

from considerations similar to the infrared RNG and for which we refer to Levich (1987). By careful reversing $\grave{}$ and $\grave{}$ superscripts in (7.18)-(7.20) we reverse the procedure from being true in infrared asymptotic limit $(k, f) \rightarrow 0$ to that in ultraviolet $(k, f) \rightarrow \infty$. Hence we obtain instead of (7.18):

$$\begin{aligned}
 v^{(n)} &= \int_n^{<} \{P(\mathbf{k}) \dots P(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)})\} \times \\
 &\times \{G(\mathbf{k}, f) \dots G(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)}, f - \sum_{i=1}^{n-1} f^{(i)})\} < \{(\mathbf{q}, f^{(1)}) \dots v(\mathbf{q}^{(n)}, f^{(n)})\} \times \\
 &\times v(\mathbf{k} - \mathbf{q}, f - f') \dots v(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)}, f - \sum_{i=1}^{n-1} f^{(i)})\} >^{F<} = \\
 &= \int_n^{<} \Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)}) \delta(\sum_{i=1}^{n-1} \mathbf{q}^{(i)}) \delta(\sum_{i=1}^{n-1} f^{(i)}) \times \\
 &\times \{P(\mathbf{k}) \dots P(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)})\} \{G(\mathbf{k}, f) \dots G(\mathbf{k} - \sum_{i=1}^{n-1} \mathbf{q}^{(i)}, f - \sum_{i=1}^{n-1} f^{(i)})\} \times \\
 &\times v^{>}(\mathbf{k}, f), \tag{8.40}
 \end{aligned}$$

where:

$$\begin{aligned}
 \Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)}) \delta(\sum_{i=1}^{n-1} \mathbf{q}^{(i)}) \delta(\sum_{i=1}^{n-1} f^{(i)}) &= < \mathbf{v}(\mathbf{q}, f) \dots \mathbf{v}(\mathbf{q}^{(n)}, f^{(n)}) >^{F<} ; \\
 \int_n^{<} &= \int \prod_{i=1}^n dq^{(n)} df^{(n)} \tag{8.41}
 \end{aligned}$$

and in the zeroth order by reversing the operations $\grave{}$ and $\grave{}$ instead of (7.20) we have:

$$v^{>}(k > k_0 e^l, f) = GF^{>}(k > k_0 e^l, f). \tag{8.42}$$

From now on the RNG procedure is very different from the one in Section 6 because we are concerned with the opposite ultraviolet limit

and it is therefore the internal wavenumbers $q^{(n)} \ll k$ and where appropriate should be set to zero to the leading order of k_0/k . Usually it would be the next step of the RNG perturbation expansion to factorize $\langle \mathbf{v}(\mathbf{q}, f) \dots \mathbf{v}(\mathbf{q}^{(n)}, f^{(n)}) \rangle^{F<}$ as a product of the pair correlation functions using the zeroth approximation (8.42). But this would only cloud the picture in this case. Let us push the scaling reasoning one step further. Namely just consider the fact that $\Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)})$ is a scaling homogeneous function so that:

$$\begin{aligned} & \Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)}) = \\ & = \Psi\{(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, \kappa_1(q^{(1)} \dots q^{(n)})f^{(1)} \dots q^{(n)}) \dots \kappa_n(q^{(1)} \dots q^{(n)})f^{(n)}\} \end{aligned} \tag{8.43}$$

and:

$$[\kappa_n(q^{(1)} \dots q^{(n)})] = -z. \tag{8.44}$$

It is now not difficult to realize that to the leading order in powers of k_0/k :

$$\begin{aligned} v^{(n)} &= v_{(0)}^{(0)} + O(k_0/k) = \int_n^{<} \Psi(\mathbf{q}^{(1)} \dots \mathbf{q}^{(n)}, f^{(1)} \dots f^{(n)}) \times \\ & \times \delta\left(\sum_{i=1}^{n-1} \mathbf{q}^{(i)}\right) \delta\left(\sum_{i=1}^{n-1} f^{(i)}\right) \{P(\mathbf{k})^n \{G(\mathbf{k}, f)\}^n v^>(k, f) + O(k_0/k)\}. \end{aligned} \tag{8.45}$$

Indeed, we have to pass to the new variables of integration $\pi_i = \kappa_i(q^{(1)} \dots q^{(n)})f^{(i)}$ and assuming the integration over them is bounded expand G and P in powers of $q^{(i)}/k$ and $f^{(i)}/f \propto q^z/k^z$.

Considering now carefully the expression, which is (8.39) with no irrelevant at this point $Q^>\{v^>\}$ and to the leading order in powers of k_0/k :

$$\begin{aligned} & v^>(k > k_0 e^l, f) = GF^>(k > k_0 e^l, f) + \\ & + (-i\lambda/2)GP \int_l^{<} \langle v(\mathbf{q}, f)v(\mathbf{k}, f) \rangle^{F<} + \sum_{n=1}^{\infty} (-i\lambda/2)^{n+1} v_{(0)}^{(n)}. \end{aligned} \tag{8.46}$$

Inspecting carefully the r.h.s. of (8.46) one can observe, in conjunction with the definition of $v_{(0)}^{(n)}$ from (8.45) and using the zeroth order

approximation (8.42) for the second term in (8.46), that it is a geometric progression and can be summed up to all orders with the result (Levich, 1980 and Levich, 1987):⁵⁹

$$v^>(k > k_0 e^l, f) = G^{ren'} F_i^>, \quad (8.47)$$

where:

$$G^{ren'} = \langle \{-if - k^z + (i/3)\lambda(\mathbf{k} \cdot \mathbf{u})\}^{-1} \rangle^{F^<} = G^{ren} \{f - \lambda/3(\mathbf{k} \cdot \mathbf{u})\}, \quad (8.48)$$

where G^{ren} is given by (8.31) with the self-energy term (8.33) and:

$$\mathbf{u} = \mathbf{u}(l) = \int_{k_0}^{<} G(q, f) F^<(q, f). \quad (8.49)$$

The last can be interpreted as a sweeping velocity of large scale velocity harmonics that we are trying to take account for shell by shell in *k-space*. Fairly remarkably all the leading perturbation expansion terms result in a peculiar renormalization of the Green function that must be analyzed. In the first place it is necessary to compare (8.48) with (7.21) to see a profound difference between the modifications of the self-energy parts in the Green functions for two different asymptotic limits, $(k, f) \rightarrow 0$ and $(k, f) \rightarrow \infty$.⁶⁰ While in the former case the corrections to Σ were all just a renormalization of the νk^2 viscous term in the latter, to all orders of the leading singularities in the ultraviolet limit, the correction is a shift in f , it is not dissipative. The meaning of $\mathbf{u} = \mathbf{u}(l)$ is clear enough as the largest scales eddies velocity that from the dimensional considerations is generally of order $L^{1/3}$.⁶¹ It looks like the whole renormalization is the frequency shift by the amount corresponding to convection of smaller eddies by the largest ones. But this is not entirely correct.^q Because $\mathbf{u} = \mathbf{u}(l)$

⁵⁹Actually the terms proportional to odd powers of λ terms are zero after the averaging, but the summation should be done inside the averaging operator $\langle \rangle^{F^<}$.

⁶⁰More precisely from symmetry considerations the summation of the leading terms in the geometric progression should be $G^{ren} \{f - \lambda/3(\mathbf{k} \cdot \mathbf{u})\} + G^{ren} \{f + \lambda/3(\mathbf{k} \cdot \mathbf{u})\}/2$ but this does not change anything.

⁶¹The parameter l is arbitrary. But usually it is chosen infinitesimal. To avoid confusion in this case it should be understood that $u(ml)$ is the largest scale velocity after many iterations of the same steps $m \gg 1$. Or alternatively one keeps l small but finite.

and because of this *l-dependence* after $m \rightarrow \infty$ iterations becomes a function of \mathbf{k} .

This is not the end of the story because we have to consider the higher order terms $Q^>\{v^>\}$. Without repeating cumbersome calculations it should be clear that exactly as we were able to extract the leading singularities and sum them up to all orders into the renormalized Green function in front of the bare forcing $F_i^>$ the same can be done with the whole nonlinear term:

$$P \int^> v^>(\mathbf{k}-\mathbf{q}, f-f') v^>(\mathbf{q}, f) \tag{8.50}$$

substituting $F_i^>$ in (8.47). Finally the equation for $v^>(\mathbf{k}, f)$ will look as follows:

$$v^>(k > k_0 e^l, f) = G^{ren'} F_i^> + \\ + P G^{ren'} \int^> v^>(\mathbf{k}-\mathbf{q}, f-f') v^>(\mathbf{q}, f') + Q'\{v^>\}, \tag{8.51}$$

where $Q'\{v^>\}$ is the sum of all higher orders nonlinearities in powers of $v^>$ and k_0/k that were not accounted for in the summation. So the first step of RNG perturbation theory was achieved and the resulting renormalized equations remain invariant to all orders in the coupling parameter and to leading order k_0/k with renormalized Green function (8.48), but in different Fourier space span. The next step of RNG is rescaling to return to the previous *k-space* span $k_0 < k < \infty$. This is done like in Section 6 (step 3), but reversing the direction in *k-space* and substituting the lowest wavenumber infrared cutoff k_0 instead of ultraviolet cutoff k_d . We rescale as follows:

$$k' = k e^{-l}, f' = f e^{-zl}, \tag{8.52}$$

so that to return to the original span in Fourier space $k_0 < k' < \infty$. Now it is convenient to redefine again $k' = k$ and $f' = f$ but remembering that both are the rescaled values. While doing the rescaling in (8.51) we keep the coefficient in the left side equal to unity to keep the equation invariant. Then we obtain the following:

$$v^>(k, f) = G^{ren'}(l)F_i^>(l) + \lambda(l)PG^{ren'}(l) \int_{k_0}^> v^>(\mathbf{k}-\mathbf{q}, f-f')v^>(\mathbf{q}, f') + Q'\{v^>; l\} \tag{8.53}$$

where:

$$\lambda \rightarrow \lambda(l) = \lambda exp[-\{3/2z - 1 + (y - D)/2\}l]. \tag{8.54}$$

The renormalization of λ except for the minus sign in the exponent, appearing because the rescaling is done in the opposite direction in $k=space$, looks exactly the same as in (7.24). But the final result will be very different. What happens after rescaling in (8.53) is that the Green function becomes as follows:

$$G^{ren'}(l) = \langle \{i(fe^{zl} \pm \lambda(l)(\mathbf{k} \cdot \mathbf{u})e^l + k^z e^{zl})\} \rangle^{-1}. \tag{8.55}$$

It is now $\lambda(l)$ instead of λ in (8.54) because after this is what the coupling constant becomes after one more iteration of the same steps of RNG. The preservation of scaling then requires that:

$$\lambda(l)e^l = e^{zl}. \tag{8.56}$$

Using (8.54) for $\lambda(l)$ and we obtain the following relation:

$$-\{3/2z - 1 + (y - D)/2\} + 1 - z = 0. \tag{8.57}$$

Using the definition (8.38) for z we obtain:

$$-(y - D + \mu)/2 + 1 - 2/3 - \mu/3 = 0. \tag{8.58}$$

From (8.28), (8.58) and restoring $D = 3$ follow the main result of this section:

$$\begin{aligned} \mu &= 0.5, \\ D_F &= 2.5. \end{aligned} \tag{8.59}$$

Let us look explicitly at what happens with the coupling parameter after many iterations $m \rightarrow \infty$. Choosing $lim_{m \rightarrow \infty, l \rightarrow 0} e^{ml} = (k/k_0)$, similarly to what was done before (see the derivation of (5.8) and (7.28) makes λ an explicit function of k (the logarithmic dependence on the infrared cutoff k_0 is neglected):

$$\lambda^{ren} = \lambda(ml) \rightarrow \lambda(k) \propto k^{-(3/2z-1+(y-D)/2)} = k^{(D-y-\mu)/2} =$$

$$= k^{(D_F - y - \mu)/2} = k^{-\mu/3} = k^{-1/6}, \quad (8.60)$$

exactly as was expected above in (8.29). The reduction of the coupling parameter naturally can be interpreted as the reduction of the interaction vertex $P(\mathbf{k}) \propto k$:

$$P(\mathbf{k}) \propto k \rightarrow P(\mathbf{k})^{ren} \propto k^{1-\mu/3} = k^z = k^{5/6}. \quad (8.61)$$

This is actually what the eddy viscosity becomes. With reference to (8.33) we conclude that:

$$[\Sigma] = [P(\mathbf{k})^{ren}] = 2/3 + \mu/3 = 5/6;$$

$$\nu_{eddy} \propto k^{-4/3+\mu/3} = k^{-7/6} \quad (8.62)$$

and restoring the dimensional scale dependence:

$$\nu_{eddy} \propto k^{-4/3}(kL)^{\mu/3} = k^{-7/6}L^{1/6}. \quad (8.63)$$

This is in contrast with the K41 eddy viscosity $\nu_{eddy}^{K41} \propto k^{-4/3}$ defined in (3.34). The real eddy viscosity is much larger than K41 eddy viscosity in the ultraviolet sub-range. Assuming that by the order of magnitude the scaling can be extended up to k_d we obtain $\nu_{eddy}(k_d)/\nu_{eddy}^{K41}(k_d) \approx Re^{1/8} \rightarrow \infty$. But the new eddy viscosity is not necessarily the dissipative one, as the renormalized self-energy part Σ is not. In fact it is compensated by the forcing with the result that the energy spectrum is K41. What it means is that the renormalized self-energy (actually frequency) acquires what can be called a propagating part that is dominant as far as the interaction is concerned but does not lead directly to dissipation. The dissipative part of self-energy appears as the next lower order term in powers of $(kL)^{-\mu/3}$, i.e., of order $\sim \lambda^{ren} k(kL)^{6-\mu/3} \propto k^{2/3}$.

It is necessary now to show that all the nonlinear couplings in (8.51) are also convergent and hence irrelevant. This was done in Levich (1987) and we refer to this paper for the details. It should be pointed out that the asymptotic convergence requires bringing non-universal corrections to the coupling constant. This point we consider briefly below.

Indeed, the asymptotic theory above was developed with logarithmic accuracy. If we look with attention at (8.56) and (8.55) it can be noticed that the sweeping velocity $u(l)$ defined by (8.49) is considered

l independent, i.e. $[u(l)] = 0$. In fact however $u(l) \propto l$, for $l \rightarrow 0$. Since after many iterations $\lim_{m \rightarrow \infty, l \rightarrow 0} ml \approx \ln k/k_0$ it means that for the convergence of the RNG perturbation expansion the coupling parameter and the frequency exponent z , as well as the eddy viscosity should acquire non-universal logarithmic correction. Since the actual parameter of expansion is $\lambda k u/f$ it should be (Levich, 1987):

$$\begin{aligned} \lambda(k) &\propto (kL)^{-\mu/3} (\ln k/k_0)^{-1/2}, \\ [f] &= 2/3 + \mu/3 + \ln(\ln k/k_0)^{1/2}, \\ \nu_{eddy} &\propto k^{2/3} (kL)^{\mu/3} (\ln k/k_0)^{1/2}, \\ \mu &= 0.5. \end{aligned} \tag{8.64}$$

The non-universal corrections are small but ideologically important underlining a different origin of the K41 spectrum from the one postulated in the K41 theory.

The relations (8.64) plus the relation (8.28) constitute the total solution of the problem as it was posed above. What should be pointed out immediately is that this solution is not a scaling solution of the Navier-Stokes equations because it does not satisfy the scale invariant set of conditions (8.65). This would not be such a bad thing, since as we know the real intermittent solutions of the Navier-Stokes equations strictly are not naively scale invariant in the above sense, if not for the fact that we argued that for \mathbf{v}_{reg} there should be a solution that is due for a quasi-Gaussian field and this is why the energy spectrum should be K41. And now we conclude that anyway the dynamical exponent $z \neq z^{K41} = 2/3$ and consequently the solution is not scale invariant. So the question is what was the special reason to believe that the energy spectrum K41 is in any way special? This is indeed a subtle point and it will be discussed a number of times. But let us notice the following. The characteristic time scale that was obtained and is in correspondence with the eddy viscosity (8.64) in the asymptotic limit, $kL \rightarrow \infty; Re \rightarrow \infty$, tends to zero by comparison with the K41 characteristic time scale, $\Delta t(k)/\Delta t^{K41}(k) \sim (kL)^{-1/6} \sim Re(k)^{-1/8}$. Simply in dimensionless units $K = k/k_d$, while the characteristic eddy $l_d = k_d^{-1}$ size turnover time $\Delta t^{K41}(k_d) \rightarrow 1$, the time $\Delta t(k_d) \sim Re^{-1/8} \rightarrow 0$. In other words the new time scale has a totally different meaning by comparison

with the usual eddy turnover time of K41 theory. This is a characteristic time of the nonlocal interactions between the eddies of very different sizes that is actually a dominant interaction. But this interaction when averaged over a finite time period, bearing always in mind that $Re \rightarrow \infty$, cancels out for certain "observable" 3D quantities. Apparently the non-local interactions are not dissipative in nature, as shall be discussed below, and in this sense can be called *virtual* (Levich, 1987). In reality the nonlocal effects are of course observable, but they are confined to a fractal sub-domain whose volume tends to zero or to high frequency dependent quantities. These observable effects are in the intermittency effects and structures and not the properties of the quasi-Gaussian field \mathbf{v}_{reg} . It is admittedly a strange reasoning for classical physics in terms of *virtual* effects, but it seems compelling in the asymptotic regime $Re \rightarrow \infty$. Further analysis will be furnished in the next section.

The solution corresponding to (8.64) is of course a scaling one as far as the Euler equations are concerned, but the smooth matching with the viscous terms typical for K41 theory is not possible as a result of the strong nonlocal interactions taking place on virtual time scales. Nevertheless, for the physical quantities defined in terms of \mathbf{v}_{reg} and averaged over a turnover time scale the matching should remain smooth as in the K41 theory.

Finally it is necessary to note that except of the above qualitative reasoning on the quasi-Gaussian character of \mathbf{v}_{reg} it remains conspicuous that there are no intrinsic indications in the model of randomly stirred fluid that would single out the choice of the parameters $y = D - \mu/3$ and $z = 2/3 + \mu/3$ with the subsequent fixation of the K41 energy spectrum. This uncertainty bothered the pioneers of this model and of RNG approach to turbulence from the very beginning while they considered the infrared RNG limit. Similar uncertainty remains with the ultraviolet RNG. In fact the only bound that can be gathered in the procedure considered above the relation (8.57) connecting the allowed values of the two basic parameters in the problem y and z and hence defining the whole family of solutions and the subsequent energy spectra on one hand and the corresponding fractal dimensions on the other. The discussion of this subtle issue will be continued below in Section 9 and Endnote s.

9 Dynamical Theory 3: Functional Integral Analysis and Interpretation of the Results

Let us analyze in more details the results of the present theory that we shall call for reference NT, a new theory.

9.1 Fractal dimension and the number of independent velocity harmonics.

The Eq. (8.59) shows that the fractal dimension of the sub-domain with reduced nonlinear coupling is $D_F = 2.5$. This is a sub-domain that is actually turbulence itself because most probably all the fluid flow in 3D space is generated by the vorticity field in this sub-domain. Accordingly the number of Fourier flow harmonics to describe the flow is:

$$N_{mode}^{ren} \approx (k_d/k_0)^3 (k_d/k_0)^{-\mu} = (k_d/k_0)^{5/2} = Re^{15/8}, \quad (9.1)$$

where the definition (2.16) was used. As expected the real number of independent velocity harmonics is much less than in K41 theory given by (2.22): $N_{mode}^{ren} \ll N_{mode}$. Although the above derivations did not use explicitly the helical nature of the sub-domain with reduced nonlinear coupling there is no other conceivable scenario but **BCC** to explain it. It is rather the other way around so that it is the discovery of **BCC** that allowed the bold assumptions in this section to be made for derivation of (8.59). Most of the N_{mode} turbulence velocity harmonics in the ultraviolet range are locked up in the helicity fluctuations.

9.2 The K41 spectrum in the inertial range.

The reality is that apparently even in the inertial range the energy spectrum is not exactly K41. This is because of the non-universal logarithmic correction (8.64). Even though we chose the K41 spectrum as a starting "free solution" the RNG perturbation theory generated logarithmic corrections.⁶² Indeed, the corresponding energy spectrum is now:

$$E(k) \propto k^{-5/3} (\ln k/k_0)^{-1/2}. \quad (9.2)$$

⁶²Such logarithmic corrections are typical for RNG developed asymptotically free ultraviolet solutions in quantum field theories. Here the solution is of course more complicated since the starting differential equations and the subsequent functional integral representations are more complicated

The logarithmic correction can be behind two facts that are quoted in Mininni et.al. (2008b). The first one is that even when the inertial range is identified in DNS the energy spectrum there has a slightly steeper slope than that for $-5/3$ power law. The second fact is the value of the constant in front of the K41 spectrum. It decreases no doubt with the increase of the Reynolds number of the simulations. Clearly from (9.2) it would follow that this "constant" is in fact asymptotically Reynolds number dependent. But of course DNS over a wider range and much higher Reynolds numbers DNS should be carried out for definitive conclusions.

9.3 The K41 spectrum, inertial range and the buffer zone.

The origin of the K41 spectrum and the inertial range are also quite different from the K41 theory.⁶³ Apart from the fact that the K41 spectrum is formed by a different mechanism, actually by a small sub-domain of intensive vorticity in physical space, the inertial range in k -space where it is likely to be true at least approximately, is effectively different. Although in the ultraviolet RNG theory developed above there is no upper cutoff it is clear that there must be a matching in k -space with the range where the viscous dissipation is dominant. In the K41 theory the inertial range was (2.21) and actually extended to k_d . However NT indicates that that the inertial range is likely to be shorter; to be sure for the high values of for which one expects the inertial range to exist at all. Most of the harmonics are locked up now in **BCC** and the number of independent harmonics is:

$$k'_d \sim k_d(k_0/k_d)^{-\mu/3} \sim k_d Re^{-1/8}. \quad (9.3)$$

It should be added that the definition of k_d itself is independent of K41 theory. It just follows from the condition (3.25) that is an exact consequence of the Navier-Stokes equations and thus remains intact. But substituting the K41 spectrum into (3.25) and carry out the integration up to $k \leq k_d$ would be wrong in a way. Because in the interval $k'_d < k < k_d$ the energy spectrum is not K41. But it does not really matter if our attitude to (3.25) is as just defining certain value

⁶³It is reminded that K41 spectrum does not necessarily mean the full K41 theory.

in the interval of wavenumbers in *k-space* that are much higher than the inertial range wavenumbers. In order to preserve the global value of enstrophy $\Omega = \langle \omega^2 \rangle$ it is necessary that most of enstrophy and energy dissipation is located in the part of *k-space* that is more ultra-violet than the inertial range. What emerges is the energy spectrum structure that is more complicated than in the K41 theory. In the latter the inertial range smoothly merges with the viscous dissipation range for which the energy spectrum is exponentially decreasing with *k*. In the present theory there should be an intermediate buffer zone connecting the inertial range with the K41 spectrum and the viscous sub-range $k \geq k_d$, with the exponentially fast decreasing spectrum, in which the viscosity effects and the nonlinear coupling are of the same order.

It should be noted that in the boundary layer (BL) turbulence the existence of an empirical buffer zone that separates the universal Prandtl logarithmic mean velocity profile and the viscous sub-range, all in physical space of course, is well known and fundamental. It is in the buffer zone where most of vortical CS are generated and most of energy is dissipated. There is a well known and far reaching relation between the logarithmic mean velocity profile in physical space for BL turbulence and the inertial range K41 spectrum in *k-space* (e.g., Levich, 1996). It is conjectured that the buffer zone in *k-space* has much similar in nature with the buffer zone in physical space for BL turbulence. I shall return to BL turbulence, the one most importance in practical applications in the next section. In the meantime it is suggested that in HIT most of energy dissipation and enstrophy birth are located not in the K41 part of the spectrum but in a more ultraviolet part of *k-space* that I call the *buffer zone* by analogy with BL turbulence.

9.4 Analysis of the functional integral representation (8.13) in the inertial range.

Let us go back to the functional integral representation and look how the results (8.59)-(8.62) fit in. Consider again the generating functional (8.22) let us substitute the results (8.59)-(8.62) into it and try to estimate the contributions of various terms in the sense of mean field approximation, i.e., approximating the functional in the

exponent in (8.22) by the mean:

$$\langle \int d^D k d^D f \Phi^{-1} \{ |\mathbf{X}|^2 + \langle\langle |\mathbf{X}|^2 \rangle\rangle \} \rangle. \quad (9.4)$$

In other words we shall assume that (8.59)-(8.62) are the most probable flow realizations dominant in their contribution to the total functional (8.22). Since the *singular* contribution is of the same order of magnitude as the *regular* one by virtue of (8.24) it is enough to estimate $\langle \int d^D k d^D f \Phi^{-1} |\mathbf{X}_{reg}|^2 \rangle$. There are three types of terms: bilinear *BL*, nonlinear *NL* and proportional to viscosity $V(k)$. The meaning of the mean field approximation now would be that we consider only the most contributing velocity field realizations that we assume to be the ones resulting in (8.59)-(8.62). The bilinear terms are:

$$BL(k) = \int_{k_0}^k d^D k' d^D f \Phi^{-1} f^2 \langle |\mathbf{v}(\mathbf{k}, f)|^2 \rangle, \quad (9.5)$$

where the integration is in a slice of *k-space* in the inertial range with $k \rightarrow \infty$. The nonlinear terms are (with tensor indices and irrelevant for the scaling analysis constant factors omitted):

$$\begin{aligned} NL(k) &= \\ &= \int_{k_0}^k d^D k' d^D f \Phi^{-1} [\lambda^{ren}]^2(k) P(k)^2 \langle \left| \int_{k_0}^k d^D k'' d^D f' v(\mathbf{k}-\mathbf{q}, f-f') v(\mathbf{q}, f') \right|^2 \rangle \end{aligned} \quad (9.6)$$

And the viscous terms are:

$$V = \nu^2 \int_{k_0}^k d^D k' \Phi^{-1} \langle |\mathbf{v}(\mathbf{k}, f)|^2 \rangle d^D f. \quad (9.7)$$

The cross terms in the integrand in the functional (8.22) are neglected. Their contribution can be easily shown to be of the same order as from the nonlinear terms that were picked up for the analysis. Now we must estimate what are the scaling powers of *BL* and *NL*, i.e. to calculate $[BL(k)]$ and $[NL(k)]$. The calculation should be done for two cases, the K41 theory and the New Theory (NT). Let us start from *BL*. The powers counting yields:

$$[BL(k)] = \int_{k_0}^k d^D k' d^D f \Phi^{-1} f^2 \langle |\mathbf{v}(\mathbf{k}, f)|^2 \rangle =$$

$$= D + [f] + [\Phi^{-1}] + 2[f] + [|\mathbf{v}(\mathbf{k}, f)|^2] = 0, \quad (9.8)$$

where the following relations were used:

$$\begin{aligned} [\Phi^{-1}] &= D - \mu/3; \\ [f] &= 2/3 + \mu/3; \end{aligned} \quad (9.9)$$

$$\begin{aligned} \langle |\mathbf{v}(\mathbf{k}, f)|^2 \rangle &= [E(\mathbf{k}, f)] + [\delta(\mathbf{k}) + \delta(f)] = [E(k)] - [f] - 2 - D - [f] = \\ &= -5/3 - (D + 2) - 2[f]. \end{aligned}$$

It is noticed that as it should be, $[BL(k)] = 0$, for both K41 theory, i.e. for $\mu = 0$ and for any $\mu \neq 0$. The meaning of this analysis is that the contribution of turbulent realizations corresponding to the K41 spectrum is indeed satisfactory as far as the BL terms are concerned since with a logarithmic accuracy it does not depend on any of the cutoffs, either k_0 or k_d . But since BL correspond to the linear term in the Navier-Stokes equations this result is trivial. The $NL(k)$ terms are the ones that are important.

The situation is different with $NL(k)$. The calculation of $[NL(k)]$ of course cannot be carried out exactly because it would be equivalent in a sense to the calculation of the functional integral itself. In reality this can be done only by RNG perturbation expansion that is quite equivalent to the perturbative solution of the model equations (8.30). But with certain assumptions an estimate can be done for $NL(k)$ as it is defined in (9.6). Let us rewrite the (9.6) by doing the reverse Fourier transformation of the nonlinear part of $X'_i(\mathbf{k}, f)$ in (8.14) in the following equivalent manner:

$$\begin{aligned} NL(k) &= \int_{k_0}^k \Phi(\mathbf{k})^{-1} d^D \int \langle \mathbf{J}(\mathbf{r}, 0) \mathbf{J}(\mathbf{r}', t) \rangle e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} d^D r d^D r' dt = \\ &= \int \Phi(\mathbf{k})^{-1} d^D \int \langle \mathbf{J}(0, 0) \mathbf{J}(\mathbf{R}, t) \rangle e^{i\mathbf{k} \cdot \mathbf{R}} d^D \mathbf{r} d^D \mathbf{r}' dt, \end{aligned} \quad (9.10)$$

where the use of homogeneity allows the change of integration variables $\mathbf{r} \rightarrow \mathbf{R} = \mathbf{r} - \mathbf{r}'$, $(-\infty < R < r)$ and $\mathbf{J} = \mathbf{J}\{\mathbf{v}\}$ is the corresponding nonlinear terms in physical space, e.g., in (8.2). Let us consider the ultraviolet limit $k \rightarrow \infty$ while keeping $\mathbf{k} \cdot \mathbf{R} \sim 1$. Let us make an assumption concerning the correlator: $\langle \mathbf{J}(0, 0) \cdot \mathbf{J}(\mathbf{R}, t) \rangle$. First of all we omit the pressure gradient potential part of the coupling and

leave only the solenoidal part, $\mathbf{J}' = \mathbf{v} \times \omega$, since this is the important one for forming turbulence structure and generating vorticity. The second and more substantial assumption is that the correlator $\langle \mathbf{J}'(0, 0) \cdot \mathbf{J}'(\mathbf{R}, t) \rangle$ can be factorized as:

$$\begin{aligned} & \langle \mathbf{J}(0, 0) \cdot \mathbf{J}(\mathbf{R}, t) \rangle = \\ & = \langle \mathbf{v}(0, 0) \cdot \mathbf{v}(\mathbf{R}, t) \rangle \langle \omega(\mathbf{R}, t) \rangle \langle \sin\alpha(0)\sin\alpha(\mathbf{R}, t) \rangle. \end{aligned} \quad (9.11)$$

This is an implicit assumption that the fastest time correlations created by the nonlinear coupling are the phase correlations and they control and balance the nonlinear interactions.

It is noted that the correlator $\langle \mathbf{J}'(0, 0) \cdot \mathbf{J}'(\mathbf{R}, t) \rangle$ has relation with the correlator describing the fluctuations of helicity $\phi(\mathbf{r})\phi(\mathbf{r} + \delta\mathbf{r})$ introduced in (5.18). Indeed, if the helicity correlators are anomalously large at small separation scales due to the alignment of \mathbf{v} and ω in helical sub-domains this would mean the corresponding reduction of $\langle \mathbf{J}(0, 0) \cdot \mathbf{J}(\mathbf{R}, t) \rangle$ in these sub-domains due to the alignment. In difference to the local in physical space interrelation between $\mathbf{v} \times \omega$ and $\mathbf{v} \cdot \Omega$ in the Navier-Stokes equations in the functional integral representation the global, non-local nature of helicity fluctuations enters naturally.

The separation of the amplitudes of velocity and vorticity in (9.11) is still possible because the velocity field is primarily determined by the large scales of order L and the vorticity field is primarily determined by the small scales l_d and these two are uncoupled except from the helicity related phase coherence that is accounted for by the correlator $\langle \sin\alpha(0)\sin\alpha(\mathbf{R}, t) \rangle$.

With all the assumptions made we can now estimate the scaling power of $NL(k)$. We have:

$$\begin{aligned} [NL(k)] &= [\Phi^{-1}] + D + [\langle \mathbf{v}(0, 0) \cdot \mathbf{v}(\mathbf{R}, t) \rangle] + [\langle \omega(0, 0) \cdot \omega(\mathbf{R}, t) \rangle] + \\ &+ [\langle \sin\alpha(0)\sin\alpha(\mathbf{R}, t) \rangle] - 2D - [dt] + [e^{i\mathbf{k} \cdot \mathbf{R}}] = \\ &= (y - D) - [dt] + [\langle \mathbf{v}(0, 0) \cdot \mathbf{v}(\mathbf{R}, t) \rangle] + [\langle \omega(0, 0) \cdot \omega(\mathbf{R}, t) \rangle] + \\ &+ [\langle \sin\alpha(0)\sin\alpha(\mathbf{R}, t) \rangle], \end{aligned} \quad (9.12)$$

where we took into account that $[e^{i\mathbf{k} \cdot \mathbf{R}}] = 0$ and $[d^D \mathbf{r}] = [d^D \mathbf{R}] = -D$. Let us consider the remaining terms above term by term. Considering the expression (3.14) for the velocity correlation function

corresponding to the K41 spectrum it is clear that the leading contribution would be $O(L^{2/3})$ and hence $[\langle \mathbf{v}(0,0) \cdot \mathbf{v}(\mathbf{R},t) \rangle] = 0$ to the leading order in powers of the Reynolds number.⁶⁴ At the same time $[\langle \omega(0,0) \cdot \omega(\mathbf{R},t) \rangle] = [R^{-4/3} = 4/3]$.

Consider first what happens if the K41 theory is accepted. Then apparently $\mu = 0$, the correlation time Δt^{K41} scales as $[dt] = -2/3$ and $y = D$. As far as the K41 theory is concerned there is no phase coherence, there are no **BCC** and the $Pdf(\mathbf{v} \cdot \omega / |v||\omega|)$ is flat. This is to say that $[\sin\alpha(0)\sin\alpha(\mathbf{R},t) \rangle] = 0$ for the time interval $t < \Delta t^{K41}$ and hence:

$$\begin{aligned} & [\langle \mathbf{J}'(0,0) \cdot \mathbf{J}'(\mathbf{R},t) \rangle^{K41}] = \\ & = [\langle \mathbf{v}(0,0) \cdot \mathbf{v}(\mathbf{R},t) \rangle] + [\langle \omega(0,0) \cdot \omega(\mathbf{R},t) \rangle] = 4/3. \end{aligned} \quad (9.13)$$

This can be interpreted as applying the quasi-Gaussian assumption to the basic correlator $\langle \mathbf{J}'(0,0) \cdot \mathbf{J}'(\mathbf{R},t) \rangle$.

Summing up together all the estimates above we arrive at the following:

$$[NL(k)]^{K41} = 2/3, \quad (9.14)$$

or equivalently:

$$NL(k)^{K41} \sim (L/l)^{2/3} = Re(k)^{1/2} \rightarrow \infty. \quad (9.15)$$

Clearly this is not a satisfactory result. In the language of the perturbation theories when the functional integral is calculated by expansion in powers of $NL(k)$ the (9.15) would mean an unrenormalizable perturbation theory with each next order term growing as integer powers of large scale or the Reynolds number, i.e., like $(L^{2/3})^n$.

But in NT the situation is totally different. In this case we have from (9.9) $y = D - \mu/3$ and $[dt] = -2/3 - \mu/3$. And now we suggest that due to the helicity phase and spatial coherence:

$$\langle \sin\alpha(0)\sin\alpha(\mathbf{R},t) \rangle = \lambda^{ren2}(R) \propto \lambda^{ren2}(k) \propto k^{-2\mu/3} \propto Re(k)^{-\mu/2}, \quad (9.16)$$

with $\lambda^{ren2}(k)$ given by (8.60). Hence it is explicitly asserted here that the reduction of the coupling constant is a result of helicity coherence

⁶⁴It is reminded that $Re \rightarrow \infty$ and this can if any of the three parameters, L, ν_L , or ν are set respectively as tending to ∞ or 0. Therefore, for instance, $O(L^{2/3}) = O(L^{2/3}/l_d^{2/3}) = O(Re^{1/2})$, where the relation (2.16) is used.

that is growing with the growth of wavenumbers or the corresponding decreasing of separation scales. In the ultraviolet limit $k \rightarrow \infty$ the helical fluctuations asymptotically tend to form Beltrami cells with extreme alignment. The subsequent singular factor $k^{-\mu/3}$ in the inverse "temperature" Φ^{-1} and the same $k^{-\mu/3}$ factor in dt give an additional factor $\propto k^{-2/3} \sim Re(k)^{-\mu/2}$. Altogether we obtain a natural coupling reduction factor $Re^{-\mu} = Re^{-1/2}$. It should be pointed out that the phase coherence (9.16) holds only for a short correlation time determined by the inverse of the typical frequency given by the scaling exponent (8.62). This frequency and the related eddy viscosity (8.63) are large by comparison with the K41 typical values, as was previously discussed. What it means is that the Beltrami like helical fluctuations of opposite sign exist only on a short time scale tending to zero in the limit $Re \rightarrow \infty$. The fluctuations are virtual in this sense. As was explained previously (in the Foreword and Sections 5, 8) the Beltrami cells of opposite sign either live only short life in much of the volume, or become stable only when their total volume is a diminutive fraction of the flow domain, asymptotically a fractal sub-domain and as a compensation the amplitudes of the velocity and vorticity fields are large (see Endnote s for discussion). In fact the velocity field inside the fractal **BCC** can be estimated as:

$$v_{sing} \sim L^{1/6} l^{1/6}, \tag{9.17}$$

instead of $v_{reg} \sim l^{1/3}$. Altogether the helicity phase coherence furnishes the required small parameter $(kL)^{-2\mu/3} \rightarrow Re^{-\mu} = Re^{-0.5}$ that is needed for the convergence of the nonlinear coupling in the functional integral representation, or equivalently the asymptotic convergence of the RNG perturbation theory developed in Section 7. This convergence occurs for the unique value of $\mu = 0.5$ as was derived in Section 7. Finally we obtain:

$$[NL(k)] = [BL(k)] = 0. \tag{9.18}$$

The results of this sub-section should be seen as to some extent independent, mathematically speaking at least, confirmation of the results of Section 8. It should be noted that all the asymptotic results above were obtained with logarithmic accuracy and disregard possible correcting factors of order $\{ln Re\}^x$. Still such factors may be relevant

and will be discussed below. Let us analyze now the viscous terms $V(k)$.^s

9.5 Energy spectrum in the buffer zone and skewness.

Is there anything that can be said based on NT about the energy spectrum $E(k)$ in the *buffer zone*? The point is that it is necessary to match the solution in the inertial range with the viscous dissipation range solution. In K41 theory this is an easy thing to do. The K41 eddy viscosity in the inertial range (3.34) is such that it matches naturally the molecular viscosity at $k = k_d$ i.e., K41 spectrum and the exponent $z = 2/3$ that corresponds to the eddy viscosity (3.34) (and of course $y = 3$), together satisfy the general scaling solution conditions (7.1). On the other hand the solution (8.62) that was obtained in Section 8 and has been reconfirmed in the preceding subsection 9.4 just now, the same K41 spectrum but $z = 2/3 + \mu/3 = 5/6$, do not satisfy the scaling conditions (7.1). As was pointed out in Section 8, the solution (8.62) is not really a scaling solution of the Navier-Stokes equations. At the same time this solution remains of course a scaling solution of the Euler inviscid equations. And now we should find a way to match this solution in the inertial range with a solution in the buffer zone where the viscous terms are of the same order as the nonlinear coupling. But the matching will not be smooth and easy now as it was for K41 theory in Section 5 above.

It was asserted in Levich (1987) that $E(k)$ in the buffer zone should have an excess of energy over what it would have been if the spectrum was K41 all the way up to k_d , as is effectively assumed in K41 theory. In other words it was asserted that $E(k)$ should have a flatter than the K41 slope somewhere in this range outside of the inertial range as it is redefined in (9.3). This can be deduced from the global matching of contributions from the nonlinear coupling term with that from the *buffer zone and viscous range*. Let us show how one can do this matching directly from the functional integral representation.

In accordance with (8.62) the nonlinear coupling generates the eddy viscosity in excess of the K41 eddy viscosity in the ultraviolet sub-range in k -space. This eddy viscosity terms should be somehow matched with the molecular viscosity terms. In particular the av-

eraged global contribution calculated for typical realizations of turbulence coming from the viscous terms should be the same order of magnitude, with logarithmic accuracy, that the contribution from the eddy viscosity terms, or which is the same the contribution from the nonlinear coupling terms. In the scaling language it means that $[V(k)] = 0$. It is tacitly assumed that scaling is still applicable in the buffer zone and this is of course an assumption. More generally it can be claimed however that globally $V(k > k'_d)$ should be Re independent to be of the same order as NL . It is suggested that the characteristic time scale in the buffer zone for the viscous dynamics is determined by viscosity ν , i.e. $\Delta t \sim \nu k^2$, which for $k \leq k_d$ is of order $\Delta t = \Delta t^{K41} \sim k^{-2/3} = R^{2/3}$, that is the time scale coincident with the K41 time scale (2.15). The correlations here are trivial between $curl\omega$ at two points decaying with time and have nothing to do with the complex correlations between helicity fluctuations as we analyzed previously for $NL(k)$. Let us rewrite $V(k)$ similarly to what was done with $NL(k)$ above as follows:

$$\begin{aligned}
 V(k) &= \nu^2 \int_{k_0}^k k^4 \Phi^{-1} \langle |\mathbf{v}(\mathbf{k}, f)|^2 \rangle d^D df = \\
 &= \nu^2 \int_{k_0}^k \Phi^{-1}(k) d^D k \int \langle curl\omega(0, 0) curl(\mathbf{R}t) e^{i\mathbf{k}\cdot\mathbf{R}} d^D \mathbf{r} d^R dt^{K41} \rangle,
 \end{aligned}
 \tag{9.19}$$

where $\mathbf{R} = \mathbf{r}-\mathbf{r}'$, $(-\infty < R < r)$ and it is assumed that the typical time of correlations for the correlator $\langle curl\omega(0, 0) curl\omega(\mathbf{R}, t) \rangle$ is the above defined Δt^{K41} , so that $[dt^{K41}] = 2/3$. At the same time the typical correlation time for the nonlinear coupling correlator in $NL(k)$ in the buffer zone remains the same $[dt] = 2/3 + \mu/3$, since in physical space we are firmly inside the **BCC** sub-domain. The temperature factor is the same for all the terms $BL(k)$, $NL(k)$ and $V(k)$. Assuming that the integrands in $NL(k)$ and $V(k)$ still can be approximated by scaling we must equalize the scaling powers for the both integrands. Also the matching with the assumed $[dt^{K41}] = 2/3$ in the region $k \leq k_d$ it should be $\nu \sim k^{-4/3}$. It is reminded that for $k \approx k_d$ the relation $\nu \sim k_d^{-4/3}$ is a definition of k_d given by (3.30) independently of what is actually the spectrum in this region. After

some power counting we obtain now the following:

$$[\langle \text{curl}\omega(0,0) \cdot \text{curl}\omega(\mathbf{R},t) \rangle] = 10/3 + \mu/3 = 10/3 + 1/6 = 7/2. \quad (9.20)$$

The exponent 10/3 is the K41 exponent for the correlator

$$\langle \text{curl}\omega(0,0) \cdot \text{curl}\omega(\mathbf{R},t) \rangle$$

that can be also revealingly written as follows:

$$\begin{aligned} S(R) &= \langle \text{curl}\omega(0,0) \cdot \text{curl}\omega(\mathbf{R},t) \rangle = \\ &= \Delta^2 \langle \mathbf{v}(0,0) \cdot \mathbf{v}(\mathbf{R},t) \rangle \propto R^{-10/3-\mu/3} = R^{-7/2}. \end{aligned} \quad (9.21)$$

The correction $\mu/3 = 1/6$ is a correction due to Beltramization. In consequence the correction in (9.20) implies a correction in the velocity correlation function and subsequently in the energy spectrum in the *buffer zone*.

The spectrum must have an excess of energy in the buffer zone by comparison with K41 power law. Let us consider this in detail. The little singular correction to the correlator (9.21) as compared with K41 theory means by continuity that the statistical moment at one point, actually at $R \approx l_d \rightarrow 0$, would also have a singular correction to the K41, i.e., Reynolds dependent factor as follows:

$$\begin{aligned} S(l_d)/S(l_d)^{K41} &\approx S(0)/S(0)^{K41} = \\ &= \langle (\text{curl}\omega)^2 \rangle / \langle \text{curl}\omega^2 \rangle^{K41} \propto Re^{\mu/4}. \end{aligned} \quad (9.22)$$

But using the relation (3.32) we obtain for the dimensionless skewness:

$$\begin{aligned} S &= - \langle (\text{curl}\omega)^2 \rangle / \langle (\text{curl}\omega)^2 \rangle^{K41} = \\ &= \nu \int k^4 E(k) dk / (\int k^2 E(k) dk)^{3/2} \propto Re^{1/8} = Re^{0.125}. \end{aligned} \quad (9.23)$$

This may be possible for instance if $E(k)$ is anomalous in the buffer zone with a flatter slope of $E(k)$ by comparison with the K41 spectrum. If scaling is assumed in the buffer zone then the slope exponent should be $-5/3 + 1/6 = -3/2$. And thus the spectrum would be:

$$E(k)_{BZ} \propto k^{-3/2}. \quad (9.24)$$

Comparison indicates that this spectrum fits well the DNS data in **Fig. 14** from Mininni, et. al. (2008). It should be cautioned though that the DNS was still carried out for relatively low Reynolds numbers.

Note that $\nu \langle \omega^2 \rangle = \epsilon$ and despite the singular corrections in (9.20)–vz (9.23) it of course remains that:

$$\langle \omega^2 \rangle = \lim_{R \rightarrow 0, t \rightarrow 0} \Delta \langle \mathbf{v}(0, 0) \cdot \mathbf{v}(\mathbf{R}, t) \rangle \propto L/l_d \approx Re^{3/4}. \quad (9.25)$$

The experiment and numerical data indicate that skewness is not constant as is implied by the K41 theory, but a slowly growing function of the Reynolds number.^t The existing data does not allow however a precise determination of the exponent. The one in (9.23) is small enough to be compatible with the existing data, but DNS with much higher values of Re than the ones considered in contemporary studies are needed for conclusive comparison.

Skewness by the relations (1.20), (3.27), (3.28) and (3.32) is connected directly with the Lagrangian vorticity stretching and enstrophy generation. Since skewness is determined fully by the processes in the *buffer zone* of k -space it means that the vorticity stretching occurs predominantly in the *buffer zone* as well and anomalously intensive as compared with what it would have been if turbulence was adequately described by the K41 theory. In physical space all active dynamics occurs in the midst of **BCC** which itself is formed as a result of the most basic processes in turbulence dynamics: vorticity stretching and subsequent breaks and reconnections, most probably at the fractal interfaces of helical fluctuations, by viscous effects.

Another quantity with which skewness is connected is flatness of the velocity field which in its turn in HIT is the normalized mean square measure of fluctuations of the energy dissipation rate:

$$\delta = \langle \epsilon^2 \rangle / \langle \epsilon \rangle^2 = \langle (\partial_x v_x)^4 \rangle / \langle (\partial_x v_x)^2 \rangle^2. \quad (9.26)$$

A rigorous Betchov inequality connects the values of S and δ as follows:

$$S \leq (2\sqrt{21})\delta^{1/2}. \quad (9.27)$$

Together the relations (5.2) and (9.23)–(9.27) result in the following lower bound for the intermittency exponent of the energy dissipation

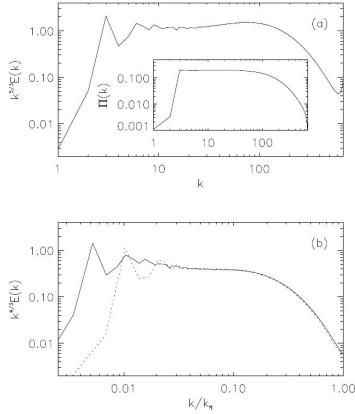


Figure 14: Shows the DNS energy spectrum from the cited work of Mininni, et.al. (2008a). The spectrum is compensated by $k^{5/3}$ factor in the upper figure and by $k^{4/3}$ factor in the lower. In the lower figure the wavenumber is normalized by the viscous wavenumber that is designated here K_η , which is the same as k_d defined by relation (3.31), i.e., $k_\eta = k_d$. It can be seen from the upper figure that the K41 spectrum is present approximately and in a very short range of wavenumbers and then clearly the slope flattens up. However the compensation by $k^{4/3}$ in the lower figure is also not perfect. The spectrum (9.24) would fit better in this post inertial range *buffer zone*. The DNS was carried out for a flow with $Re \approx 10^4$ and claimed good resolution for up and over $k_\eta = k_d$. Although this is not a sufficiently high value of Re for final conclusions it is still about the same as in many laboratory experiments. The usually reported K41 spectrum in these experiments looks better than in DNS. But as was emphasized before measurements in turbulence are extremely difficult and in particular the spectrum is not directly measured as a rule but is inferred from related quantities using certain assumptions. Therefore it well may be that the DNS results may be the ones to favor. The inset in the upper figure shows the energy transfer flux and seems constant with good accuracy in both the short inertial range and in the part of the *buffer zone*.

fluctuations (scaling is assumed):

$$\mu_\epsilon \geq 1/3 \approx 0.33. \quad (9.28)$$

This inequality clearly fits well the median experimental data that usually quotes $\mu_\epsilon \approx 0.4$, but again the experiment is insufficiently accurate and will remain such in the foreseeable future. It is likely that we will have to wait for much more powerful DNS for the final determination of S and μ_ϵ . Nevertheless, both the results (9.23) and (9.28) are realistic and inspires certain optimism.

Summarizing we can state that the prediction of the existence of the *buffer zone* in k -space in which the nonlinear coupling is still essential but at the same time the viscous terms are equally contributing is rewarding. Apparently while in physical space all the important dynamical processes are taking place primarily in **BCC**. Concomitantly in k -space it is in the *buffer zone* where most of enstrophy is generated and located and most energy is dissipated. In this sense if we push the analogy a bit further it can be asserted that while the inertial range in k -space is in correspondence with the universal logarithmic profile range in BL turbulence, likewise the *buffer zone* in k -space is in relation with the *buffer zone* of BL turbulence in physical space. This analogy will be pushed further below where it will be asserted that the helicity fluctuations and Beltramization determine the BL turbulence structure as much as they do in HIT.

9.6 Energy spectrum in (\mathbf{k}, \mathbf{f}) -space.

Although in NT the energy spectrum $E(k)$ remains K41 the full spectrum $E(k, f)$ is not what it is in K41 theory. Because $[f] = 2/3 + \mu/3$ and in consequence the general scaling form for the spectrum is:

$$\begin{aligned} E(k, f) &= C \langle \epsilon^{2/3} \rangle k^{-7/3-\mu/3} L^{-\mu/3} \phi(f/\langle \epsilon^{1/3} \rangle k^{2/3+\mu/3} L^{\mu/3}) = \\ &= C \langle \epsilon^{2/3} \rangle k^{-5/2} L^{-1/6} \phi(f/\langle \epsilon^{1/3} \rangle k^{5/6} L^{1/6}) \end{aligned} \quad (9.29)$$

This is instead of:

$$E(k, f)^{K41} = C \langle \epsilon^{2/3} \rangle k^{-7/3} \phi(f/\langle \epsilon^{1/3} \rangle k^{2/3}). \quad (9.30)$$

Integrating (9.29) over k we obtain:

$$E(f) \propto f^{-9/5}. \quad (9.31)$$

Measurements of turbulent spectra for the last half century have been so confusing and imprecise that it would be naive to compare (9.31) with experiment. It suffices to say that it is definitely compatible with the available experimental data.⁶⁵ But this is not enough to feel satisfied at this time.

9.7 High order velocity field structure functions (5.3).

NT does not provide answers for the intermittency exponents of the velocity structure functions that are often chosen, probably because of their relative simplicity, for experimental studies in turbulent flows and whose anomalous exponents are seen by many in relation with the multifractal structure (see also Endnote s). On the other hand the basic intermittency understood as singular corrections to skewness and flatness factor come out naturally in a quantitative manner from the theory on a fundamental level as a phenomenon that is likely to be the property of subviscous buffer zone in Fourier space while **BCC** in physical space. In fact the existence of **BCC** makes it impossible for intermittency not to exist. On the other hand the very nature of statistical description through the correlators and moments of turbulent fields seems limited as was mentioned previously in Section 5. Indeed, the higher order is the correlation function or a single space point single time moment of a turbulent quantity the bigger is the contribution of **BCC**. The same is true for high order derivatives of the velocity field since they are all determined by the high wavenumbers range from the buffer zone. In other words the high order statistical quantities would just reflect the inner structure

⁶⁵There is a certain sense of frustration when one sieves through the experimental data for the energy spectra. The early data was comprehensively reviewed in Monin and Yaglom (1975). As was emphasized throughout this paper the experiment in turbulence is extremely difficult. It remains to wait for more computational power to be able to increase the Reynolds numbers of simulations by at least one order of magnitude. Unfortunately, this is not going to happen soon since it would mean almost three orders of magnitude increase in computational requirements (see (2.22)). As far as the frequency spectrum is concerned the long time discussions are whether the spectrum is $f^{-5/3}$ or f^{-2} . The arguments seem to be well resolved if the spectrum (9.31) is accepted with the NT exponent $-9/5$ lying exactly between the contested exponents $-5/3$ and -2 . For meteorological analysis of the data see, e. g., comprehensive Radkevitch et al. (2007), Liley et al.(2008) and Radkevitch et al.(2008).

of **BCC**. But **BCC** consist of completely coherent flow patterns that most likely are totally unsuitable for statistical description. I conjecture that this is the dynamical reason of algebraic tails for large deviations from the mean values in the probability distribution functions of turbulent fields that seems to manifest in geophysical turbulent flows with very high Reynolds numbers (e.g., Schertzer and Lovejoy, 1985a and 1985b; Radkevitch, et.al., 2008).

As was mentioned in Section 5 such algebraic tails were not observed in laboratory experiments or DNS. But all these have been made for relatively low values of the Reynolds number. Nevertheless, there is a consistent indication that for high enough orders $N > 6$ the exponents of the velocity structure functions become linear functions, i.e., $\mu_\nu(n) \propto n$. The linear dependence on (generally non-integer) n would be indeed the only possible for the probability distribution functions with algebraic tails (see, e.g., Levich, 1987, for a simple explanation). An approximate value $n \approx 6$ may be singled out because on certain qualitative approximation, it can be suggested that $\Delta \mathbf{v}^6 / r^2 \approx \epsilon(r)$ and hence $\langle \Delta \mathbf{v}^6 \rangle / r^2 \approx \langle \epsilon^2 \rangle$. In other words the structure function of this order or close to this order is likely to exist in the limit $Re \rightarrow \infty$. But it seems that even this is not the case in certain atmospheric turbulence situations (Radkevitch, et.al., 2007).

9.8 Non-local versus local interactions: energy transfer in \mathbf{k} -space.

There seems no doubt that in a long range of wavenumbers the energy transfer in turbulence is constant and substantially local in k -space, i.e. subject to the triads' requirement (7.29). DNS indicates the constant flux range extending through all of the inertial range and ostensibly the buffer zone (Mininni, et.al., 2008a and 2008b). If the energy flux would not be constant the whole scaling approach to developed turbulence would be in doubt.

On the other hand the very fact of appearance of the $L = k_0^{-1}$ dependence in the characteristic parameters (8.64) leaves no doubt that the interactions in turbulence are predominantly non-local. In fact in a qualitative way it can be said that the eddies with a scale l are coupled most strongly with the eddies having the velocity defined

by (9.17) and scales $\sim l^{1-\mu/3} L^{\mu/3} = l^{5/6} L^{1/6}$.⁶⁶ From the viewpoint of Fourier space **BCC** is a high wavenumber cluster. In physical space, judging on **Fig. 1** and **Fig. 3**, it consists of asymmetric helical vortices having laterally all sizes from the inertial range and smaller and stretched up to a fraction of L in one direction. Thus the phase coherence is definitely most important for strongly disparate scales of motion locked through this coherence with each other.

But the non-local interactions do not mean nonlocal energy transfer, as was commented above. One notion "phase coherence" resolves the issue. For the local energy transfer between the velocity harmonics in k -space to exist it is necessary that the non-local phase coherence exists as well. The nonlocality is not a correction over the principle of local energy transfer but a necessary condition for the local energy transfer to exist (Levich, et.al., 1991). However it is likely that this statement is correct with logarithmic accuracy and hence the K41 spectrum is correct only with logarithmic accuracy. The approach with growing to local in Fourier space energy flux is slow, as is clearly demonstrated by Mininni, et. al. (2008b).

The detailed analysis will be carried out elsewhere and we confine the present considerations to some general qualitative reasoning. The point is that if we consider the energy transfer expression in the model equations (8.30) it would consist of several terms each of them proportional to $L^{\mu/3} = L^{1/6}$ and thus is dominated by the nonlocal interactions. This is easy to understand from the fact that renormalized eddy viscosity (8.62) is proportional to the same $L^{\mu/3} = L^{1/6}$ factor. On the other hand when the energy spectrum is calculated there is a cancelation of likewise large terms and as a result we obtain the L independent K41 spectrum. It was suggested in Levich (1987) that the virtual frequency of nonlocal interactions (8.64) is a propagating mode and not a dissipative one. The helical fluctuations at all scales live and die during a short time span of correlations defined above, or stabilize in a diminutive volume corresponding to **BCC**. The energy swings forth and back between all the scales in conjunction with these fluctuations, but all this takes place on a short time scale. Therefore the systematic energy flux from larger scales to the smaller ones takes place on the usual scaling time scale $dt^{K41} \sim k^{-2/3}$.

⁶⁶It is a good place to check that the "volume" of these eddies scales as the cube of the scale, *propto* $(l^{5/6})^3 = l^{5/2}$, i. e., as an object of dimension $D_F = 2.5$.

If these assumptions are made than scaling analysis in the spirit of NT would be indeed compatible with the locality of the energy flux in k -space. If to approximate the transfer term with the eddy viscosity term and the forcing at the same time then it is clear that the eddy viscosity term does not lead to a systematic energy transfer and this is why the K41 spectrum is a solution. In other words the nonlocal interaction is not dissipative but rather propagating and its single mission is creating phase correlations in helicity fluctuations (and subsequent intermittency). When averaged over K41 time interval the energy transfer from nonlocal interactions would be tending to zero. Indeed, averaging over K41 time furnishes a scaling factor $\sim dt^{nonlocal}/dt^{K41} = dt^{nonlocal}/dt^{local} \propto k^{-\mu/3}$. Additionally the coupling constant reduction yields another factor $\propto k^{-\mu/3}$. Together they compensate the integral scale $L^{1/3}$ factor in the expression for the energy flux transfer that would appear from the direct scaling analysis of the transfer term that is not shown here. However this is an asymptotic reduction effect and the ratio of nonlocal and local energy transfer contributions would most likely tend to zero logarithmically slow as a function of $Re \rightarrow \infty$.^u

10 Dynamical Theory 4: BCC in Wall Bounded Turbulence and Possibility of Turbulence Control

There is no experimental or numerical data sufficient to develop quantitative NT for even the simplest wall bounded turbulent flows. The main conjecture that will be advanced here is that certain properties that were discussed for HIT remain invariant independently of the flow geometry. In the first place such feature as **BCC** and its fractal dimension. In other words it is conjectured that **BCC** remain the sinews of turbulent flow field whatever is the geometry and boundary conditions.

There are few solid facts at this time to support this claim. Primarily I rely on the fundamental universality principles in their most general sense that have guided research in turbulence for the last one hundred years and common sense. Indeed, the general similarity between the wall turbulence and small scale properties of HIT is

obvious to anyone. That distance from the wall in many ways plays the role similar to the inverse wavenumber in HIT in near to dissipation region is also obvious. As the small scales of HIT generate spontaneously the helicity fluctuations necessary to organize **BCC** for the purpose of controlling the nonlinear coupling and subsequent smooth flow of energy on their way to viscous oblivion, so I conjecture that the wall region does the same in wall bounded flows. The wall bounded turbulence is of course much more complicated because while these processes are taking place in physical place they are also occurring in the conjugate Fourier space. This is confusing and it is most likely that the Fourier space is not the right one to introduce in wall bounded flows. In fact the anisotropy of wall bounded flows allows to use other than Fourier orthogonal sets of eigen functions for decomposition of the velocity field and extracting the most relevant and coherent features. Such is the Karhunen-Loeve decomposition that has been applied extensively for the analysis of turbulent channel flow and pipe flows (e.g., Sirovich and Zhou, 1994, Rajaei, et.al., 1994, Webber, et.al., 2002, Dugdale, et.al., 2007). We refer to these papers for details but just mention that this decomposition is not as fundamental as Fourier decomposition and is data dependent and must be carried out on the basis of available experimental or DNS data. In fact it is a way to sample out the most typical and hopefully most important features of the data. It is a general conclusion of many researchers using this methodology or its modifications and analyzing DNS and experimental data that near to wall turbulence has some specific features.

It seems that turbulence in the wall region is dominated by elongated streamwise oriented streamwise vortical structures called the streaks. The streaks streamwise length L_{rolls} expressed in the wall units $\delta = \nu/\nu^*$ is significant, probably of the order of 1000δ and longer. The streaks have a peculiar internal structure. First of wall they carry the dominant part of turbulent field component energy. This velocity in the streaks strongly favors streamwise direction with the streamwise velocity component u considerably bigger than the other two velocity components (see **Fig. 18**). The roll structures carry most of the total turbulent energy contained in the fluctuating part of the velocity field. In this sense they are like the large eddies in HIT. From time to time over a distance in spanwise direc-

tion z the streamwise velocity of the streaks changes sign. Roughly speaking two adjacent streaks are two streamwise counter-flows. The distance between such two counter-flows is usually quoted in literature as about wall units, $L_{streak} \sim (100 \pm 20)\delta$. But in reality this streak spacing seems to be growing with the distance from the wall y^+ . Whether the streak spacing is function of Re_τ is not clear from the available data. The necessity to have counter-flows is obvious because the streamwise fluctuating velocity should be close to zero when averaged over a big enough spatial flow domain.

The streaks themselves apparently consist of smaller structures that can be likened to counter-rotating rolls. The counter-rotating rolls seemingly have a dominant vorticity aligned along x streamwise direction. From time to time a rotating roll vortex changes the rotation sense and becomes counter-roll. This is also obvious since the total streamwise preferred fluctuating vorticity should also become zero (nearly) when averaged over a big enough spatial flow domain. The roll spacing also seems to be growing linearly with the distance from the wall y^+ . But it is much smaller than the streak spacing, of order $L_{roll} \sim (30 \pm 5)\delta$. There is no reliable data on a possible dependence on Re_τ .

The existence of such streaks with roll structures embedded is supported by DNS and measurements, but both lacking precision and definitiveness. Nevertheless, let us consider the **Figs. 15** from Rajaei, et.al. (1995). The authors carried out measurements in water channel flow with moderate Reynolds number as already briefly described above in Section 6, **Fig. 12**. In **Fig. 15** the authors plot the spanwise two point velocity correlation function for the streamwise velocity component, $Re_{uu}(z)$ for two distances from the wall for reference. One can clearly see the negative minima of $Re_{uu}(z)$ at different z^+ for two different distances from the wall y^+ . The location of the minima corresponds to $1/2L_{streak}$. For the smaller wall distance $y^+ = 16.7$ the streak spacing is about, $L_{streak} \approx 160$, which is bigger than the ones usually quoted from DNS carried out for lower Reynolds numbers. But this does not imply that a conclusion on Re_τ dependence can be made at this stage. For a large distance from the wall the negative minima is quite broad and not as well delineated. Still the negative minimum at $L_{streak}/2 \approx 300$ seems discernible.

In **Fig. 15** the authors plot the normal to the wall velocity com-

ponent two point correlation function $R_{\nu\nu'}(z)$. At two different distances from the wall, the farther is deep inside the logarithmic profile universal region $R_{\nu\nu'}(z)$ shows well discernible negative minima. This minima signifies the change of sign of the normal velocity component v by the rolls, albeit for the spacing Δz^+ different for every y . The behaviors of the two correlation functions $R_{uu'}(z)$ and $R_{\nu\nu'}(z)$ are very strongly indicative of the existence and general structure of the streaks and rolls as described above.

Two remarks should be made. In reality of course the streaks of and rolls are not strictly periodic and are most probably quasi-randomly spaced. Nevertheless, the negative correlations seems definite and therefore the conclusion of certain typical dimensions for both the streaks and the rolls. On the other hand their position in spanwise direction can be even random while the minima in the correlation functions will remain. The presence of these minima does not indicate by themselves the spatial coherence.

Another remark has to do with the fact that the sense of rotation in the rolls determines the sign of streamwise vorticity ω_x and therefore we can conclude that the sign of ω_x reverses in the counter-rolls.

This is not the end of the story. From time to time the streaks and rolls meander in the spanwise direction and also in the directions normal to the walls and erupt into small-scale activity. While they meander in spanwise z^+ -*direction* they produce spanwise velocity component and spanwise vorticity component. But the relative direction of velocity and vorticity remain the same. When the streaks and rolls lift up normal to the wall they create normal velocity and vorticity components and then again the relative direction between the two does not change. It is only when the uplifted rolls erupt in small scale debris then the velocity and vorticity misalign and become random. This eruption is the phenomenon of so-called bursts and sweeps and these two probably produce most of small scale turbulent activity that is subsequently dissipated. The streamwise length of the streaks and rolls of order 1000δ is actually the length that the rolls maintain their stable state that is the typical distance between the bursts.

The bursts are ejection of the relatively slower moving at the walls fluid from the inner boundary layer into the regions of faster moving fluid in the outer layer, and the sweeps are on the contrary to the in-

rushes of faster moving fluid into the regions closer to the walls. The bursts and sweeps are manifestations of acute turbulence intermittency in the boundary layer and they contribute significantly to the momentum flux at the walls and subsequent turbulent drag. They are primarily responsible for the peaks of activity in **Fig. 13**. The alternation of bursts and sweeps is necessary by the fluid continuity.

How dynamically the streaks and rolls can break? This by the general construction of the Navier-Stokes equations can proceed only in triad interactions between the rolls mode and two other modes. This is what was analyzed in Sirovich, et. al, (1991). It was found that the roll mode, that is nearly stationary in the sense that its life time is very long, interacts with two other faster time propagating modes. It should be pointed out that the analysis was carried out using the general method of Karhunen-Loeve eigen functions decomposition.

These eigen functions are build in such a way that they form an optimal basis for decomposition of a field of interest, in this case the turbulent velocity field, in a mathematically well defined optimal way (e.g., Sirovich, 1991). The Karhunen-Loeve decomposition can be proved optimal in a rigorous mathematical sense. In the framework of Karhunen-Loeve eigenfunctions analysis the resonant triplets in wall bounded turbulence, made of rolls and pairs of propagating structures, substitute the simple non-linear wave triplets in Fourier space for HIT. In difference to Fourier or Chebyshev eigen functions decomposition that are fundamental and do not require *a priori* knowledge of the velocity field, the Karhunen-Loeve eigen functions are built based on empirical data on the field inferred from experiment or DNS.

In this sense Karhunen-Loeve decomposition is always somewhat subjective and requires detailed backward comparison with the real fields, the subjects of decomposition. The great advantage of Karhunen-Loeve decomposition on the other hand is that they capture, at least in principle, the main features of the fields with relatively small number of harmonics. If there is coherence in the field the Karhunen-Loeve decomposition is in principle an optimal tool to extract it from the flow. In this language the rolls described above have been extracted from channel flow turbulence as degenerate time independent (which is obviously an approximation) energetic harmonics,

while their meandering and uplifting as interaction with resonance oblique to the mean flow wave patterns forming triad interactions with the rolls and propagating with the mean flow (Sirovich, et.al., 1991), Webber, et.al., 1997). To reiterate the oblique propagating structures are non-linearly coupled with the rolls in such a manner that a roll and two propagating waves form resonant triplets. These oblique propagating structures considered as the spanwise meandering and uplifting of the rolls can be interpreted as the typical modes of instability of the rolls. But in the same time the coupling of two oblique waves can generate a roll. This necessitates the propagating harmonics to carry sufficient energy as well. This duality is the essence of the nonlinear coupling between the rolls and the propagating wave like structures.

It should be pointed out that the rolls and oblique patterns are statistically defined and become apparent and delineated only when a sufficient number of realizations of turbulent flow are considered in a properly designed statistical scheme. An important property of the wave structures is the coherent manner in which they are dynamically coupled with the rolls. It was found that if they are excluded from the statistical analysis the Reynolds stress time trace loses its intermittent character. In reality of course the streaks, rolls and oblique modes are more complicated and limited number of modes analysis that was considered by DNS of relatively low Reynolds numbers channel flows has not been yet the final word in the matter. Nevertheless, the works of Sirovich and his co-authors are perhaps the best analysis of what is happening in, at least simplest, wall bounded flows. The fact that the bursting and sweeping events are tied up with instability of the streaks and rolls that can occur only in resonant triad interactions with oblique modes and that these propagating structures are the precursors of rolls instability and by implication carriers of intermittency seems to stand on firm ground (e.g., Webber, et.al., 1997).

The triad picture of interactions is of course inevitable from the general structure of the nonlinear coupling in the Navier-Stokes equations. In the Karhunen-Loeve decomposition however the coherence of the vortical structures enters naturally to some extent, while for HIT coherence and anisotropy of **BCC** is hidden in the Fourier analysis and averaging. In HIT the averaging hides anisotropy and co-

herence and only visualization and sampling reveal them. In near to wall turbulence the mean flow orients the structures accordingly and they become much more obvious to eye and sampling analysis such as Karhunen-Loeve decomposition.

Note though that so far the streaks and propagating structures were not identified as helical structures, but this is likely a matter of time only. All the qualitative description of rolls and their meandering without losing the relative alignment between the velocity and vorticity quite strongly point to this end. Additional indications can be inferred from the following DNS results for a turbulent channel flow (Shtilman and Levich, 1995 unpublished). Helicity per unit volume in a channel flow can be written as follows:

$$H = 1/2\Delta \int \langle h_{\perp}(y, t) \rangle' dy = 1/2\Delta \int h_{\perp}(y) dy, \quad (10.1)$$

where:

$$\begin{aligned} h_{\perp}(y) &= h_{\perp}^x(y) + h_{\perp}^y(y) + h_{\perp}^z(y) = \langle h_{\perp}(y, t) \rangle' = \\ &= 1/L_x L_z \int u(x, y, z, t) \omega_x(x, y, z, t) + v(x, y, z, t) \omega_y(x, y, z, t) + \\ &\quad + w(x, y, z, t) \omega_z(x, y, z, t) \rangle' dx dz. \end{aligned} \quad (10.2)$$

The mean velocity $U(y)$ and the mean $\Omega_z(y)$ vorticity do not contribute to the mean helicity.

The quantity $h_{\perp}(y)$ therefore is helicity partially averaged over the plane (x, z) and over time. It can be compared to $H(k)$ spectrum in HIT. In the latter case we saw a nontrivial behavior of the mean helicity spectrum generated by the viscosity and nonlinear coupling in the ultraviolet range in k -space. And now we want to see the behavior of a similar quantity in the near to wall region in a channel flow. To do this we performed DNS of a channel flow with inflow-outflow boundary conditions in streamwise x -direction and periodic boundary conditions in spanwise z -direction as is usual, and the channel half width in normal to the walls y -direction corresponding to the Reynolds number (6.6), $Re_{\tau} = \Delta/\delta = 180$.⁶⁷ The ensuing steady

⁶⁷In DNS of turbulent channel flow the velocity field is represented in normal to walls inhomogeneous y direction as Chebyshev polynomials expansion. In (x, z) plane velocity is assumed having periodic boundary conditions and the usual Fourier expansion is used.

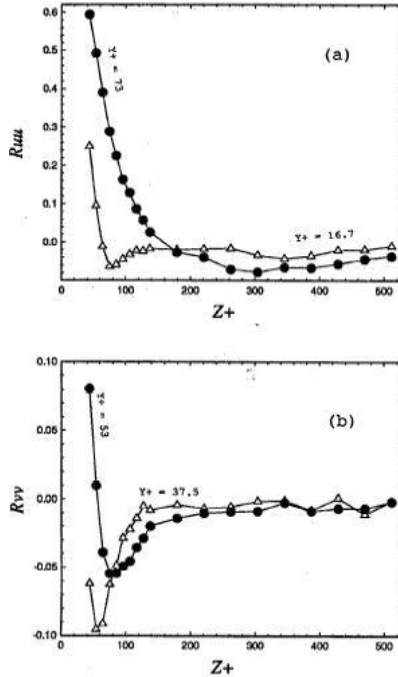


Figure 15: From Rajae, et. al (1995) they show two point spanwise correlation functions for the streamwise and normal to the wall velocity components. The negative minima correspond respectively in a and b to the mean streak spacing and the mean roll spacing in spanwise direction.

state turbulent flow was well resolved and showed the mean velocity profile and the Reynolds stress almost identical to the ones in **Fig. 11 and 12** (carried out for a slightly smaller $Re_\tau = 125$) and adequate for this still relatively low value of Re_τ .

The result for $h_\perp(y)$ inferred from this DNS is shown in **Figs. 12a and 12b**. The depicted $h_\perp(y)$ is a result of time averaging over 40 time realizations of the flow. Helicity is shown only for one half of the channel since it looks very much the same in both halves of the channel. Also it has the same sign near the walls and near the centerline regions. The dashed line shows the streamwise component $\nu\omega_x$ for comparison:

$$h_\perp^x = 1/L_x L_z \int \langle u(x, y, z, t)\omega_x(x, y, z, t) \rangle' dx dz. \quad (10.3)$$

Inspection of **Fig. 15** clearly supports the above assertion of the correlated nature of the helicity fluctuations in the wall region. This helicity coherence is quite similar to what was observed for the helicity spectrum in Fourier space in the simulations of the BigBox turbulence if we only as a thought experiment compare the mean helicity spectra at $l = k^{-1}$ near to the dissipation range in **Fig. 9** and the distance from the wall near to the viscous sublayer. Indeed, in the small $l = k^{-1}$ range the helicity spectrum in **Fig. 9** is strongly coherent and is positive for most of wavenumbers. Thus the summation over the range would lead to amplification of helicity value. The $h_\perp(y)$ shows much the same and is everywhere negative in the wall region and everywhere positive in the central part of the half channel (and negative in the other half of the channel). But the amplitudes of $h_\perp(y)$ are much higher in the wall region with well developed maximum at $y_{h_\perp^+}^+ \approx 32$. In the part of the buffer zone with most of turbulence production and dissipation on the contrary the mean helicity is quite smaller. Very close to the wall, ostensibly in the viscous sublayer, the mean helicity has the second local maximum, primarily due to the streamwise component h_\perp^x and probably has to do with primal streamwise vortices produced by wall friction. Generally in the buffer zone $h_\perp(y)$ is made up of two contributions primarily, the streamwise h_\perp^x and spanwise h_\perp^z , so that approximately $h_\perp \approx h_\perp^x + h_\perp^z$. If helicity is associated with the streaks than the spanwise component h_\perp^z would be related to the rolls meandering in

z-direction and propagating oblique modes. In general the structure of h_{\perp} seems compatible with the conjecture that turbulence in the wall region is primarily made of **BCC**. We see in **Fig. 16** and **Fig. 17** that the h_{\perp}^x qualitatively closely follows $h_{\perp}(y)$ as long as $sy^+ < 50$. This similarity of h_{\perp} and h_{\perp}^x tells something about the structure of the flow in the wall region. Indeed, consider an isolated roll. It is defined by a certain direction of large-scale u and a sense of rotation, as was described above that generates large scale ω_x , say both u and ω_x directed opposite to the flow. The small-scale contribution to velocity and vorticity vanishes when the averaging is carried out over the roll volume. In counter-rolls separated by the streak spacing the direction of u would change to become in the flow direction, so that the sum of the streamwise velocities of the streaks vanish. The direction of ω_x also reverses in the groups of rolls separated by the roll spacing, so that the total streamwise vorticity in a group of rolls would cancel as well. But there is overlap when both u and ω_x change directions at the same time. When it happens the product $u\omega_x$ remains with the same sign and thus the mean streamwise helicity component over span of two adjacent rolls is generated. The rolls meander without changing the prevailing directions relative to each other, but these would become spanwise now and hence generate h_{\perp}^z . The flow in reality is rather statistical so that neither velocity nor vorticity are exactly compensated in adjacent separate pairs of rolls or streaks. However, qualitatively the tendency is as described when the averaging is done over many rolls. These considerations indicate that the relative scale that is given by the ratio of energy and helicity $E(y)/h_{\perp}(y)$ would be of order $\geq 2L_{streak}$, which in this particular DNS turns out to be the case. The cited work of Shtilman and Levich, (1995) was not followed up and thus no detailed data on helical structure of the streaks and rolls is presently available. But I would like to express confidence that this is a matter of time only when the helical structures in the channel flow will become a subject of interest as it has happened with other turbulent flows, HIT, jets and geophysical structures.

BCC conjecture brings to mind a quite fascinating vision of wall bounded turbulence. **BCC** in the limit of very high Reynolds numbers tends to become fractal surface, multiscale and rugged. All the small scale events then can be seen as just the **BCC** fractal surface

penetrating into the outer layer turbulence and actually forming the flow profile there. This looks for the relatively low Reynolds numbers flow analysis as meandering and uplift of delineated rolls and bursts and sweeps with a semblance of time regularity. In reality the picture of events may be much more complicated and irregular, if not to use the word chaotic.

Note that the location of the helicity maximum is exactly the same as for the maximum of Reynolds stress in the particular DNS of Shtilman and Levich (1995), as can be clearly seen in **Fig. 17**:

$$y_{h_{\perp}max}^+ = y_{max}^+ \approx 32, \quad (10.4)$$

where generally for any Reynolds number y_{max}^+ as a function of Re_{τ} is given by (6.32). The conjecture that I would like to make is that generally also:

$$y_{h_{\perp}max}^+ = y_{max}^+ \approx \sqrt{Re_{\tau}/\kappa}. \quad (10.5)$$

The equality (10.4) is intrinsic to the main conjecture of this section that turbulence near to wall is made up of **BCC** and should be verified by DNS carried out over a range of Re_{τ} , which is a task for the future, but the fit (10.4) for a particular DNS used here is quite unlikely a mere coincidence.

The total helicity and the helicity of each half channel very slowly, with a typical period of four, five large eddy turnover times, change sign. When averaged over a very significant number of realizations it becomes small in clear similarity with the BigBox turbulence (see **Fig. 10**, Section 5), although the mean helicity at the walls does not appear to change practically at all as if being in a real steady state. The spontaneous helicity of the flow in the wall regions appears to be extremely durable in time. It should be reminded that the mean helicity is not at all the optimal quantity representative of **BCC**. It is the fluctuations of helicity that should be identified. Nevertheless, even the average helicity may serve as an indication, as does the mean helicity spectrum in HIT. We also note that in a way the wall bounded flows are convenient to analyze, provided that there are no limitations on resolution. Indeed, what we have analyzed above is the inner structure of **BCC**, if of course near to wall turbulence is indeed made up of **BCC**. This is much more difficult to do for **BCC** in HIT. It is also again reminded that in physical units the "width" of **BCC** and the inner wall region alike tend to zero in the limit $Re_{\tau} \rightarrow \infty$.

I would like to built up on the empirical observation (10.4). Let us estimate the volume, in the wall units, of the sub-domain up to y_{max}^+ . In accordance with (6.32) it is:

$$\begin{aligned} V_{y \leq y_{max}} &\sim (\Delta L_x L_z / \delta^2) Re_\tau^{1/2} = (\Delta L_x L_z / \delta^3) (\delta / \Delta)^{1/2} = \\ &= V_{total} (\delta / \Delta)^{1/2}, \end{aligned} \quad (10.6)$$

where V_{total} is the total flow domain volume. If this sub-domain is made up of **BCC** it means that relative to the total volume the **BCC** sub-domain volume is also $\sim (\delta / \Delta)^{1/2}$. But this is exactly the same as for the **BCC** sub-domain in HIT. In the latter case the **BCC** sub-domain volume is $\sim (l_d / L)$ and in the limit $(l_d / L)^{-1} \rightarrow \infty$ the sub-domain tends to a fractal with $D_F = 2.5$. I conjecture that this is what happens as well in wall bounded turbulence in the limit $(\delta / \Delta)^{-1} \rightarrow \infty$.

As in HIT the whole flow is created and sustained by the **BCC** sub-domain so it is most likely similar in wall bounded flows. In wall bounded flows the **BCC** from the *buffer zone* penetrates like a tongue into the outer flow as it is meandering in spanwise and normal to the wall directions. It looks, for low Reynolds numbers flows at least, as meandering in the spanwise direction rolls and their uplifting from time to time at an angle to the walls with generation of oblique patterns as a precursor of bursts and sweeps as discussed before. With the growth of Re_τ **BCC** in near to wall region must become increasingly complicated as is fit for the semi-fractal sub-domain with the maximum Beltramization lying inside the logarithmic velocity profile range. This allows an optimal flow of energy from the outer flow to the walls, as the K41 spectrum provides an optimal flow of energy towards the **buffer zone** and dissipation region in *k-space*. In HIT this is only possible with certain helicity related phase coherence as was explained in Section 5. In wall bounded turbulence it is likely to be similar and there should be strong helicity related phase coherence in place. However the phases are not in Fourier space in this case but for instance of the Karhunen-Loeve decomposition harmonics locked in correlated wavepackets. These are responsible for the non-zero mean helicity and naturally the effect is especially pronounced in the region near to the maximum of the Reynolds stress that is at the same time the location of maximal reduction of the nonlinear coupling.

It is noted that by continuity in the vicinity of extremum of the Reynolds stress the nonlinear interaction between the mean and fluctuating parts of the velocity field is relatively more reduced. Applying the Reynolds averaging to the identity (1.9) yields for the Reynolds stress:

$$\partial_k \langle v_i v_k \rangle \equiv -\epsilon_{ikl} \langle v_k \omega_l \rangle + \delta_{ik} \partial_k \langle v_l v_i \rangle / 2. \quad (10.7)$$

Therefore it seems natural due to continuity to expect that near this location the Beltramization of helical fluctuations would be maximal with the associated maximal relative reduction of the nonlinear coupling. It is also likely to be the location of a weakest compensation of the positive and negative helicity fluctuations and of subsequent peak of local mean helicity. In low Reynolds number flows as are in the scope of modern DNS this location is on the edge of the *buffer zone*, but it will move away from the buffer zone with the increase of Re_τ in accordance with (6.32).

Summarizing it is conjectured that as in HIT the nonlinear coupling is relatively reduced, better to say balanced by **BCC** in such a manner that the K41 scaling is held, similarly **BCC** balances the Reynolds stress in wall bounded flows in such a way that it is approximately constant and thus forms the universal logarithmic profile. I would like to note that **BCC** exists not only in the near to wall region. In fact the structures are everywhere so that it has sense to introduce **BCC**(**y**) concept to underlie this fact. And most likely it is this immensely complicated structure that creates the whole wall bounded turbulence. It is expected that as in the case of HIT in the limit $Re_\tau \rightarrow \infty$ the **BCC**(**y**) will be a fractal domain with maximal dimension $D_F = 2.5$.

But the **BCC** in the wall region may have particular significance for the problem of turbulence control since much of the turbulent drag comes from near to wall streaks meandering, uplifting and bursting in the *buffer zone*. Although I suspect that for truly high values of Reynolds numbers the situation may be different in the sense that the near to wall **BCC** and **BCC**(**y**) farther from the walls would not be well delineated.

It also necessary to reiterate that the boundary between the *buffer zone* and the region with universal logarithmic profile may be also not cast in stone at $y^+ = 30$, as is usually presented. Indeed, building

upon the above conjecture it would seem clear that if the distances from the wall are such that they overlap the scales interval that is the inverse of wavenumbers from the *buffer zone* in *k-space* (see Section 9) than the energy spectrum is not K41 anymore and the considerations that led in Section 6 to logarithmic velocity profile, i.e., Eqs. (6.15) and (6.16) are not true anymore.

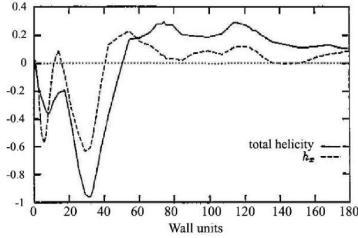


Figure 16: From Shtilman and Levich (1995), published in Levich (1996), it is a plot of the mean helicity $h_{\perp}(y)$ in half channel averaged over many time realizations and for comparison the same for the streamwise component h_{\perp}^x .

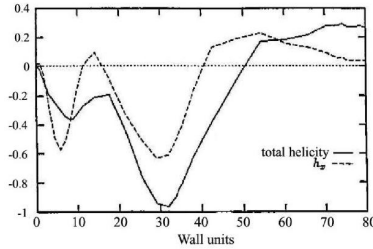


Figure 17: The same as in Fig. 15 but with more details in a part of the half channel flow up to $y^+ = 80$.

Therefore the profile in the buffer zone should be accordingly reconsidered. We shall confine the discussion of this issue for the time being to comments in the endnote below, because there are so many

things that are not understood in wall bounded turbulence that obtaining one more formula that cannot be unambiguously verified by experiment or DNS at this stage is rather premature.^v

It was conscientiously attempted and in fact established numerically for a flat channel turbulent flow that interfering by external perturbations into the turbulence dynamics, in such a way that the phases of certain harmonics, e.g., corresponding to oblique waves, are randomly mixed from time to time, results in fundamental modification of the turbulent flow and in particular in modification of the rolls structure and dynamics (Handler, et. al., 1993). The overall Reynolds stress $\langle uv \rangle$, its absolute value dominated by the rolls in the near to wall region, decreases and its maximum shifts away from the walls as is seen in **Fig. 18**, even though the energy content of the rolls does not decrease. The fluctuating velocity component $\langle u^2 \rangle^{1/2}$ decreases significantly, while the streamwise fluctuating velocity component $\langle u'^2 \rangle^{1/2}$ increases as is seen in **Fig. 19**. The characteristic spanwise size of the rolls grows as is seen by considering the shift leftward of the maximum in the spanwise energy spectrum $E(k_z)$. This maximum in the spectrum corresponds to the minimum of the correlation function $R_{uu'}$ in **Fig. 15**. The typical time span between the bursts increases. The conclusion one makes is that the random phase oblique waves are incompatible with the intrinsic coherent coupling mechanism with the rolls, in consequence the dynamics of the rolls formation and disintegration becomes anomalous, the bursts and sweeps rate diminishes and effectively the energy flux to the walls also diminishes. This last can be seen in **Fig. 18** as the shift and reduction of the rate of turbulence production. A point of potential interest is that the effectiveness of interference with turbulence dynamics not only depends on the group of modes which are chosen for phase decorrelation but also on the frequency of this decorrelation. Indeed, it appears that there is a resonance optimal time interval between the successive phase decorrelations, where the time is expressed in wall units time $T^+ = Tv^{*2}/\nu = T/(\delta v^a st)$ and T is the usual time in computational time steps. This resonance frequency of phase decorrelations corresponds to the largest effect of drag reduction. The dependence of the impact that the phases decorrelation makes on turbulent flow is further illustrated in **Fig. 20** by different shifts of the maximum of the spanwise energy spectrum $E(k)$. But

interestingly the largest effect of drag reduction does not correspond to the minimal frequency of phase decorrelations chosen in the DNS, corresponding to the interval $T_{min}^+ = 4.69$, but a smaller frequency corresponding to the interval $T^+ \approx 7$. If such resonance frequency is confirmed to be a real effect it can be of practical importance for attempts to control drag.

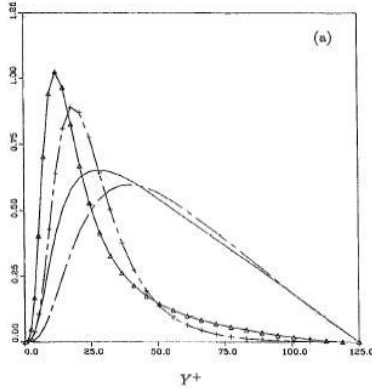


Figure 18: From Handler, et.al. (1993) it shows the transformation of the Reynolds stress and turbulence production for a specific DNS run when the phases of certain groups of velocity modes were randomized every few viscous times. The viscous time is defined as: $T^+ = T\nu^{*2}/\nu = T(\delta\nu^*)$, where T is the normal computer time in computational steps. The run corresponds to a particular $T^+ = 4.69$, Reynolds stress $\langle uv \rangle / v^{*2}$, normal turbulence: (—), drag reduced turbulence: (- - -). Turbulence production $\langle \epsilon \rangle (y) \approx 5 \langle uv \rangle dU/dy(\delta\nu^{*2})$, normal turbulence (Δ), drag-reduced turbulence (+).

The flow modification when perturbed as described above in many ways looks remarkably similar to that in the wall bounded turbulence caused by a dilute polymer solution injected in the *buffer zone*. This well known empirically and very dramatic in scope phenomenon still remains unexplained, despite a number of conflicting empirical theories. At the same time the phase randomization effect is quite

different and much bigger in magnitude by comparison with all other known empirical methods that led to admittedly meager drag reduction in developed wall bounded flows (e.g., widely discussed riblets). The turbulent drag was reduced by phase randomization by nearly 60% in similarity with the largest drag reduction achieved by injection of polymers into the *buffer zone*.

From the view point of **BCC** the effect of turbulence modification and subsequent drag reduction seems natural. Indeed, even though **BCC** occupies only a very small sub-domain in turbulent flow it is nevertheless, as we argue, is the heart of all turbulence and actually forms the flow in the whole domain. In particular if in the framework of HIT the helicity related phases $\alpha(\mathbf{k}, t)$ are randomized while in a manner similar to the one just described the rate of energy dissipation is drastically reduced (Levich, et.al., 1991; Murakami, et.al., 1992). The phase randomization is a rather clear test for turbulence coherence and its importance. As a matter of principles it allows to radically modify and probably manage turbulent flows by using minimally intrusive perturbations on natural turbulence, but in such a manner that the perturbations pinpoint **BCC**. This is like a surgery with minimal intrusion applied in precisely localized way to the site that should be incised. It should be pointed out that phase randomization that is numerically implemented on the flow Fourier harmonics corresponds in physical space to some workless forcing and in this sense is the least intrusive.^w In the wall bounded flows the interference with **BCC** would mean that in a thin layer of turbulent flow adjacent to the walls should be perturbed in a certain way that mimics the phase decorrelations in Fourier space. The possibility is that polymers do exactly this by interfering with the wall **BCC** structure and dynamics. The fluid sticks to long polymer molecules and definitely unravels and stretches them as it does with vorticity lines. The elastic effect causes feedback reaction on the vorticity lines stretching and this is what can likely disrupt the coherent mechanisms of **BCC** dynamics.

Turbulence management in wall bounded flows is a hugely important subject. Thousands of research works were devoted to it in the last 50 years. But except of empirically observed large effect of drag reduction by polymer additives there is little to report in terms of achievements. Few percent of drag here or there that may be useful

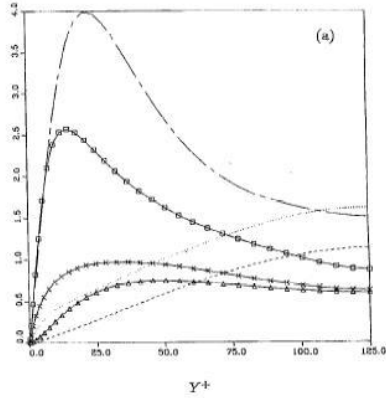


Figure 19: Root mean square fluctuating velocity components for normal and phase randomized turbulence (T^+). Normal turbulence: $u' = \sqrt{\langle (u/\nu^*)^2 \rangle}$ (\square); $v' = \sqrt{\langle (v/\nu^*)^2 \rangle}$ (\triangle); $w' = \sqrt{\langle (w/\nu^*)^2 \rangle}$ (\times). Phases randomized turbulence: u' (---); v' (----); w' (····).

in particular applications, but no general approach and few ideas. If it was not for the fact that a few drops of polymer substance cause such a tremendous effect of drag reduction one would think that turbulence cannot be tamed and drag cannot be diminished. But we do have this glaring example that proves the opposite. The phase randomization is the only instance when a theoretically applied concept demonstrated that an affect of the same magnitude as with polymer additives can be achieved in principle. It would seem that such a discovery would generate a splash of experiments and works trying to expand in this direction or on the contrary close this line of enquiry if it turns out to be wrong. For instance the effect of drag reduction of about 3% by riblets (shallow streamwise groves at the walls) has been a subject of hundreds of publications over a period of many years and the arguments are still on whether it is one percent more or less, if the effect exists at all (most probably it does).

Unfortunately this is not what has happened after the publication

of the results by Handler, et.al. (1993) and there are still very few papers that touch upon the mechanisms described here. In neutral fluids this is of course a very difficult and delicate task to implement an almost non-intrusive perturbation. It still remains unclear whether static perturbations may substitute for the dynamic perturbations that are equivalent to the phase decorrelation. Certain attempts have been made to implement the drag control by passive methods (static perturbations). Sirovich and Karlsson (1997) reported a large drag reduction for a channel flow using passive random disturbances that were supposed to mimic in some way the phase decorrelations in the flow. They were trying to implement the passive means for drag control that were described in Sirovich, et. al. (1994) and Sirovich, et. al. (1998). Unfortunately it was later disclosed by the authors that their results were wrong in part or totally as far as the measurements were done and analyzed.

The second attempt was reported in Monti, et al. (2001) trying passive means somewhat similarly to the previous research of Sirovich and Karlsson but with variable parameters of surface perturbations adjusted to the growing BL turbulence. The authors claimed substantial positive drag reduction in a certain region of BL and overall drag reduction. As far as I am aware of there were no serious attempts to validate or repudiate these results.⁶⁸

The measurements of drag reduction are excruciatingly difficult task and many attempts and plenty of experimentation and ingenuity are needed to confirm the validity of one or another result. As far as the mechanisms briefly described here are concerned these are the tasks for the future. But the most outstanding prime objective is the positive identification of **BCC** in the wall region. If this is done then it would be possible to proceed with conscientious program of interfering with **BCC** in a beneficial manner. In the meantime it remains to live with a verdict passed by Bushnell in his comprehensive "Aircraft drag reduction-a review" (2003) where he comments on the works of Handler, et.al., (1993) and Murakami, et.al., (1992), among a number of others from the category he calls 'vision', as follows: "At this point the real-time control of turbulent wall dynamics remains an

⁶⁸Both experimental programs were initiated and supported by grants from Ormat Industries Ltd. (Israel) and its subsidiary Orlev Ltd. (Israel). The work of Monti et al. (2001) was also supported by EC JRC, Ispra.

extremely interesting ‘vision’”. Hopefully the engineering community will take upon the task of testing this extremely interesting ‘vision’ along the lines outlined here.

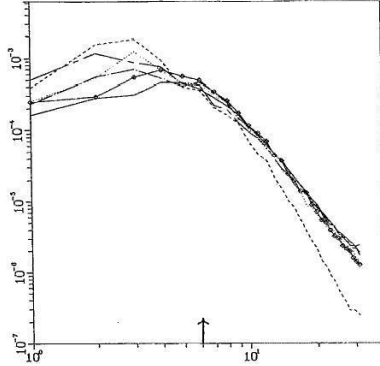


Figure 20: One-dimensional energy spectrum $E(k_z)$ at $y^+ = 15$. The solid line is the normal turbulence. There is a well defined peak indicated by the arrow and corresponding to the streamwise rolls having in this case the width $\sim 50\delta$ (wall units). The rolls are also clearly observed by visualization of near to wall turbulence (e.g., Hinze, 1975). The other lines are the plots of the spectra modified in different runs with phase randomization done for different values of T^+ , the minimal being 4.69 and the maximal 23. All modified spectra have the maximum shifted to the lower values of k_z indicating that the corresponding rolls become wider in spanwise direction as $L_{roll} \approx (k_{max})^{-1}$. Since the rolls hold most of turbulent energy (at least in near to wall region) it means that the energy content shifted to larger scales.

11 Concluding Remarks

In the essay of Lumley and Yaglom (2001), “*A Century of Turbulence*”, the two outstanding scientists in the field of turbulence wrote: “We believe it means that, even after 100 years, turbulence studies

are still in their infancy. We are naturalists, observing butterflies in the wild. We are still discovering how turbulence behaves, in many respects. We do have a crude, practical, working understanding of many turbulence phenomena but certainly nothing approaching a comprehensive theory, and nothing that will provide predictions of an accuracy demanded by designers". It is difficult not to agree with these words.

During the 100 years that Lumley and Yaglom allude to the whole scientific landscape has been dramatically transformed. Newton's vision of Universe gave way to General Relativity and Quantum Mechanics. And some leading physicists asserted that they are close to Theory of Everything. Someone, anonymously of course, hinted and caused popular stirring at creating quantum black holes in a rather ordinary looking but admittedly impressive in size collider. And some biologists assure us that except of details they know how organized life has evolved from primordial chaos by pure random selection and wish to survive. And in all this spectacle of power of scientific thought and claims we have to confess to have no clear idea how the usual storms are formed and organized, what is the origin of coherence in ubiquitous turbulence that we can observe with our naked eye every day all around. It is not that we don't know the details; on the contrary we know plenty of them. What we don't know are the very principles of organization in turbulence. Surely this glaring ignorance must make us feel slightly less arrogant as researchers.

And it is not that fewer or lesser minds worked on fundamentals of turbulence. Some great physicists and mathematicians tried their hand in this field and left little imprint for posterity. Some were successful in the sense defined by Lumley and Yaglom. They formulated certain general principles of turbulence as science discipline and great engineers developed ingenious methods for practical applications bypassing the need of fundamental understanding. But the need is still there and for the theory of turbulence not to stagnate it should leap forward to a new level of awareness. Such leap would be a comprehensive understanding of how and why from the initial chaos coherence in turbulence originates and evolves.

In this paper it is asserted that strongly correlated helical fluctuations is at least a partial answer that furnishes quite a puzzling mechanism at the core of the origin of coherence. I would like to ob-

serve that conjecture on the role that helicity fluctuations may play in turbulence was made over 25 years ago, and not only it did not die, as many conjectures in this field did over time, but on the contrary has found important experimental confirmations. This made it possible in this paper to expand the scope and ramifications of this conjecture and to develop it into real theoretical assertions and even claims.

To summarize it is asserted that the essence of all turbulent coherence are small sub-domains of nearly Beltrami flows cells with the opposite helicity sign that are clustering together and make up coherent structures. These Beltrami cells clusters, shortly **BCC**, are the relict of shortlived helical fluctuations that serve the purpose of reducing, or more correctly balancing the nonlinear coupling in the Navier-Stokes equations in such a way as to provide optimal conditions for the energy flux into the small scale harmonics, in conjugate Fourier space and/or towards the boundaries, so that energy can dissipate freely into heat. But this process necessitates that strongly coherent virtual helical fluctuations are formed that evolve into **BCC** filling a small flow sub-domain, but responsible for most of vorticity generation and energy dissipation. In fact this sub-domain can be seen as turbulence and most likely by induction creating and sustaining the whole flow domain into which it is embedded. It was claimed that asymptotically **BCC** has the leading dimension $D_F = 2.5$. But likely there are smaller dimensions sub-domains playing important dynamical role and making **BCC** a multifractal object.⁶⁹

BCC are asserted to be the building blocks in all turbulent flows, from flows in pipes to tropical hurricanes and beyond. Similar structure of coherence may be also present in MHD plasma turbulence, i.e., with the magnetic field playing the role similar to vorticity in usual turbulence, but this remains to be thoroughly clarified.

BCC may present possibilities for turbulence management, and in fact interference with **BCC** coherence could be the only fundamental way to control turbulence. If MHD turbulence has similar coherent build-up this may pave way to entirely new mechanisms for plasma turbulence management. But this is too large a subject for this pa-

⁶⁹The matching of NT with multifractality should be a subject of separate study as was already pointed out in Section 9. Certain considerations to this end are discussed in Endnote s.

per. In the last several years anomalously large helicity fluctuations and associated stabilizing reduction of the nonlinear interactions were found in various turbulent flows. From HIT to MHD turbulence in solar wind, to turbulent jets (Gavita, et. al., 2008)^x, to compressible flows turbulence (Andreopoulos, 2008) and most revealingly in geophysical structures, e.g., hurricanes, midlatitude storms and tornadoes, where the presence of anomalous helicity was established quite unambiguously. That most geophysical mesoscale structures (and their environs) are helical fluctuations and this is what gives them their relative stability as a result of reduction of the nonlinear coupling was predicted in Levich and Tzvetkov (1984 and 1985). The concept was in a more specific way advanced by Lilly (1985). An extreme concentration of helicity has been recently reported by Molinari and Vollaro (2008) from their analysis of tropical hurricane Bonnie (August, 1998). One snap-shot of Bonnie borrowed from Molinari and Vollaro (2008) is shown in **Fig. 21**. Helicity (in definition accepted in geophysical studies) is given by numbers at certain locations along with the indications of the shear wind at these locations. Without going into details it can be seen that not far from the center of Bonnie helicity values are especially high. The authors assert that the stability of the Hurricane Bonnie may have to do with extreme values of helicity in its certain parts called supercells. Similar observations were made previously for midlatitude storms and tornadoes (e.g., Fei Shiqiang and Tan Zhemin, 2001). Unfortunately the relevant observations and some other less meaningful studies, or rather musings of the last two decades on the role of helicity, often confusing mean helicity with helicity fluctuations, have not led to quantitative analysis and indeed remained primarily on the level of 19th century naturalistic observations on peculiarities of certain butterflies.⁷⁰ It is hoped that the present study will help understanding that **BCC** is not one more peculiarity of turbulence but an issue of prime significance that if confirmed further beyond doubt may re-define understanding of turbulence. It is important to analyze with as many fine details as possible the presence of **BCC** in various turbulent flows first of all in

⁷⁰I would like to remark once again that mean helicity by itself generally plays little dynamical role but it is helicity fluctuations that play the basic dynamical role as described in this paper. When researchers observe intensive helicity flow domain they should be looking for a domain with anti-helicity somewhere.

BL turbulence. If **BCC** are there, as is predicted in this paper, this would mean considerable leap in understanding turbulence coherence. The positive and universal identification of **BCC** would allow studying their inner structure and may give practical insights into ways to control turbulence, in neutral fluids and plasmas.

I would like to detour into what some would think of as a philosophical detraction, and comment on a general feature that may have bearing on non-equilibrium systems other than turbulence. A rather obvious point is that while studying certain non-equilibrium complex systems one encounters situations in which a system is injected energy by external source and this energy is relatively well organized, for instance has comparatively low number of degrees freedom and in consequence relatively low entropy, $S^{in} \leq S_0$, and subsequent to this energy injection the system seemingly by itself⁷¹ creates some inner coherent organization, while at the same time energy is eventually all dissipated out of the system but having on the whole much higher entropy $S^{out} \gg S_0$. A closest to us all example is the Earth itself seen as a system, e.g., Penrose (2005). Daily and yearly it receives solar energy. Although the solar source is thermal but primarily the energy is in ultraviolet and blue and visible red part of the spectrum. The solar energy is mainly reflected back, but some small fraction of it is absorbed and captured by various elements on Earth, oceans primarily, vegetation, etc. When captured this energy fraction facilitates amazing things, from organization of atmosphere and oceans, turbulent organization to be sure, to photosynthesis and all the way down to biological organization. Oceans use this energy in particular to organize atmospheric circulation in conjunction with its own circulation. And this is done through turbulent mechanisms. Vegetation through photosynthesis creates oxygen and this facilitates biological organization and so on. I have no knowledge of all the processes and the balances of energy re-distribution and primarily guessing using some common sense. But what is clear is that all this used energy is dissipated back into space, or rather excreted. Because even though the energy that is dissipated out into space is the same as the energy that was absorbed, nevertheless it is different in quality. It is spent energy that is irradiated as heat in primarily infrared part of the spec-

⁷¹This is in difference to say a power station that is also a useful system, a converter of energy, but works as directed by engineers.

trum. Its photons are much less energetic and the total number of photons is thus much bigger in order to have the same amount of energy. In consequence the energy coherence is much less. While energy was stripped of its coherence Earth as a system used this coherence, in particular to maintain the turbulent organization of the oceans and atmospheres circulation, cascading all the way down probably to biological organization, as vividly described by Penrose (2005).

The examples when energy flowing through non-equilibrium systems loses its coherence that is utilized by the complex systems for creating certain inner *smart* organization are many. From unicellular organisms that generally eat highly organized organic nutrients and excrete lower organization matter, while ostensibly utilizing the nutrients coherence for creating and sustaining its own organization, to formation of galaxies from uniform gas where the galaxies relative coherence is compensated by increased entropy of gravitational field.

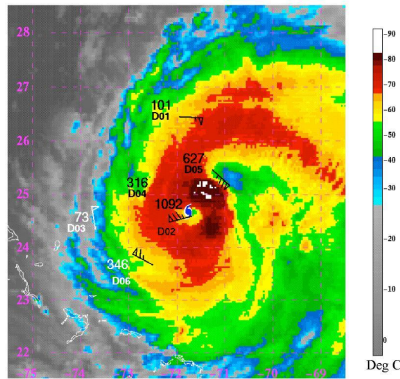


Figure 21: Field measurement of a typical tropical hurricane Bonnie made in August 2001 and analyzed by Molinari and Vollaro (2008). The values of helicity are indicated by the numbers in certain units and claimed by the authors to be exceptionally high and as stated by the authors contribute to the Bonnie stability.

It is fairly amazing to observe that the same underlying principle is apparently as basic for turbulent flows. This puts turbulence in a

league of all important smart non-equilibrium systems that can work as machines metabolizing lower entropy energy into higher entropy energy and while doing this creating and sustaining highly organized structure. Indeed, turbulence is created and fed initially by relatively low entropy sources with relatively few degrees of freedom, would it be a laminar-turbulent instability or even a random forcing that acts primarily at large scales and thus has relatively few degrees of freedom. In the end all the energy received is dissipated out of turbulence as heat having many degrees of freedom and low organization. Energy is stripped of its coherent content. But instead the flow acquires coherence through the birth of highly organized **BCC**, confined though to a small sub-domain in physical space. The similarity is striking but in the case of turbulence we can investigate in detail this process.^y It may be important to note that this organization is not what is sometimes called "self-organization". There is nothing of this kind in turbulent flows. Hurricanes and tornadoes do not self-organize, or originate from clever models. The coherent organization of turbulence is a specific phenomenon imposed by certain very particular properties of the specific and deterministic equations of motion, a conjugation of the Eulerian nonlinear dynamics, rigorous and unique for ideal continuous media, and fundamental dissipation processes that are phenomenologically matched with the ideal dynamics by the viscous term in the Navier-Stokes equations.

Maybe we are getting a glimpse of how coherence evolves in other more complex non-equilibrium dynamical systems, in an ordered and deterministic manner. Because as complex as turbulence is by itself there are many other systems around us surely more complex, for which we do not know dynamical equations that would show the reasons for and actually impose coherence development as intrinsic part of their dynamics. In this context it would be right to classify turbulence as "simplest of complex systems".⁷²

Endnotes

^a No doubt those complex local conditions of intertwined atmo-

⁷²The definition of turbulence as "simplest of complex systems" belongs to E. Tsvetkov to whom I am indebted for introducing me into the fascinating field of atmospheric turbulence.

spheric and oceanic flows are all important for the formation of particular organized geophysical events. But they are facilitators and not the underlying reasons for the very possibility of existence of coherent structures in atmospheric and ocean turbulence. A number of outstanding geophysical studies note the intrinsic global coherence of atmospheric phenomena at all scales. I will cite some that I am familiar with in what follows. But the point is even now these studies have gained only limited recognition by mainstream meteorologists despite the many observations proving their validity.

It is important to have fundamental answers to all the "whys" of the geophysical turbulent organization. For instance there should be turbulent mixing of temperature and humidity in the Earth's atmosphere for the biological species as we are to thrive. This is what the tropical storms do when entering midlatitudes, a wonderfully benign phenomenon. Not as hurricanes are portrayed in popular press - destructive element and growing menace as the emission of CO_2 in the atmosphere continues. The currents play equally important role in the global climate formation (e.g., Elliot, 2007). Recent observations (e.g., Oey, et.al., 2007) show deep interrelation between tropical hurricanes and ocean currents.

Many smaller scale "violent" atmospheric events serve towards similar goals. All these events are most probably intertwined and tuned to an amazing degree so that to maintain stable and organized global atmospheric and oceanic coherence. Consider if for one year suddenly there would be no tropical storms. The consequences would be totally disastrous, I suspect, on a global scale. Luckily we can be sure that as the season approaches the tropical storms shall form. Remarkably all geophysical observations strongly indicate that the horizontal distribution of energy in a wide range of scales probably from planetary scales to the scales of millimeters follows as a function of scale the same power law. Moreover significant segment of geophysicist argue, based on observational data, that the effective dimension of the atmosphere from this viewpoint is not $3D$, as one would think looking around, and is not $2D$ as one would expect considering huge planetary scales such that the vertical depth of the atmosphere is negligible by orders of magnitude by comparison with the horizontal ones. Comprehensive analysis of diverse experimental data resulted in determination of the leading fractal dimension of the

turbulent structures in atmosphere as $D_F \approx 2.55 \pm 0.003$, (e.g., Lovejoy, Press Release of McGill University, 2004, Lovejoy, et., al., 2008, Lilley, et.al, 2008). This is very close to, actually indistinguishable from $D_F = 2.5$ that is asserted in the present paper to be the leading fractal dimension for the dynamically active part of turbulence in all turbulent flows; isotropic or anisotropic alike. The interpretation by the above authors is quite different and motivated by the atmosphere vertical convective stratification theory of Bolgiano (1959) and Obukhov (1959), but the closeness of the numbers is worth noting.

Other little known but very illuminating example of an extraordinary coherence of geophysical patterns are the small in size by comparison with tropical storms but similar in structure hurricanes in Eastern Mediterranean bringing rain to Israel and adjacent Middle East areas with great regularity for thousands of years (Tzvetkov, 1985; Levich and Tzvetkov, 1985). Without these hurricanes much of the Middle East civilization probably would have not come around. So many seemingly disconnected factors over large territory should be in a sort of resonance for these hurricanes to form. And nevertheless with some variations the same pattern persists over historically long periods of time.

^b How turbulence originates from smooth laminar flows is a subject of immense mathematical and practical interest. It was mentioned above that big advances and profound understanding of transitional turbulence were achieved in the pioneering works on strange attractors of dynamical chaos explaining, e.g., Lorenz (1963), Ruelle and Takens (1991), Ruelle (1978) and their followers who launched the vast modern theory of dynamical chaos, the basic unpredictability of nonlinear systems. The basis for which can be traced all the way back to the works of Poincare in the beginning of the 20th century, Since practically everything in nature is strongly nonlinear, from simplest mechanical devices to planetary systems and biological species and financial markets the overwhelming philosophical concept prevailing among scientists and educated laymen has become that everything is chaotic and unpredictable. There is much truth in it as almost all researchers (and not only researchers) know from personal experience and computer simulations. The real issue that remains unanswered now is why despite this undeniable chaos and unpredictability the global world around us, as exemplified by the turbu-

lent atmosphere, is superbly stable when considered on an appropriate space and time scales. This order can be traced to systems that have nothing to do with turbulent fluids but this is not the purpose of this paper.

Most probably all routes to dynamical chaos follow the Feigenbaum's scenario of universal period doubling. There is little doubt that this is how turbulence starts in laminar flows of fluids as well. Unfortunately, or actually very fortunately for us since the world around us is stable, the understanding of the onset of turbulence does not give much of a clue to the properties of turbulence long after the point of transition to turbulence at certain critical values of $Re = Re_{critical}$. The Feigenbaum mechanism explains what happens near $Re_{critical}$. But the developed turbulence is a state when the value of the Reynolds number is far beyond the critical one for the onset of turbulence. The analogy may be drawn with phase transitions. The Wilson's RNG theory of second order phase transitions explained to us what happens in liquid He_2 , for instance, when it is cooled to the point of λ transition to superfluid state. It becomes very strongly fluctuating between the normal and superfluid state. The same happens at the critical point of all other systems near the second order phase transition. But it does not tell us what the flow properties of superfluid He_2 are for the temperatures much below the critical one.

^c Helicity is formally defined as $\int \mathbf{v} \cdot \boldsymbol{\omega} dV$, where the integration is done over the volume of a compact domain D bounded by the vorticity lines, i.e., such that the normal projection of vorticity on the boundary of D is zero: $\omega_n|_{\partial D}$. Since the helicity density $h = \mathbf{v} \cdot \boldsymbol{\omega}$ is a pseudo-scalar helicity can be either positive or negative and changes sign under mirror transformation. In mirror symmetric flows helicity is identically zero.

^d Turbulence is a multiple scale motion of fluid elements. Each scale of motion contains certain amount of the total energy of turbulent motion. The energy spectrum is the energy distribution that describes how much energy is contained in the different scale motions. In this sense it plays the same role for turbulence, a non-equilibrium phenomenon, as Maxwell or Bose-Dirac energy distributions play in equilibrium gases. But in difference to molecular motion where energy is attributed to so many molecules in turbulence the role of

molecules are played by macroscopic fluid elements of various spatial scales and life span dependent on the scale. The Kolomogorov theory says that fluid elements, or eddies as they are called, of typical scale l have typical velocity $v_l \sim l^{1/3}$. Therefore the energy distribution as a function of scale is $E(l) \sim l^{2/3}$. The largest eddies are inclusive of all the smaller ones; roughly speaking the smaller eddies are inside the bigger ones and their energy does not contribute much to the largest eddies. This picture is much more complicated than in the molecular dynamics. In Fourier space of wavenumbers $k \sim 1/l$ the energy spectrum becomes $E(k) \sim k^{-5/3}$, so that $E(l) = \int_{k \sim 1/l}^{k_d} E(k) dk \sim l^{2/3}$, where $k_d \gg 1/l_d$ is so-called dissipation cutoff wave number the meaning of which will be explained below. The energy spectrum also determines how much vorticity is contained in these different scales of motion. It turns out that although the main energy is contained in the large scale motion, as should not surprise anyone with the mundane experience of swimming in the waves or aeronautical experience. But the most vortical is the small scale motion of the order $l_d \sim k_d^{-1}$. In fact again it is well known from the experience of bathing in the waves when one is dragged under water and carried by an energetic wave. The body then is convulsed by intensive small scale vortical eddies inside the wave and especially near the bottom pulling in different directions. This mundane experience of bathers and aeronauts alike when described mathematically is a very complex and peculiar duality between the most energetic eddies and the most vortical is behind much of the richness of turbulence phenomenon.

^e It is necessary to caution that the calculation was made in the framework of a particular type of renormalization group and perturbation theory scaling analysis. In general the perturbation theories of any kind when applied directly to the Navier-Stokes equations are totally unacceptable for analytical description of turbulence, whether it is a closure or sophisticated renormalization group theory. As a field theoretical problem the Navier-Stokes equations are not renormalizable in the relevant asymptotic limit of small space and time variations, the so-called ultraviolet limit. Roughly speaking it has to do with the fact that the parameter of perturbation theory is proportional to a power of and grows indefinitely in this limit. However when certain assumptions on the nature of the flow are made then the perturbation theory solutions are possible. The correctness of

these solutions however is determined by the correctness of underlying physical assumptions. In other words a solution is as good as the assumptions that allowed the perturbation and scaling analysis to be meaningful. However this particular scaling analysis is very different from the many others that were tried before. Briefly the difference is as follows. The Kolmogorov theory implicitly postulates that the only important types of non-linear interactions between various velocity harmonics are among the local in the space of wavenumbers triads. In other words the only important are the couplings between the triads of eddy of similar size. On the other hand the interactions between the eddies of disparate sizes are postulated to be vanishing in the limit $Re \rightarrow \infty$. This assumption will be explored below in more details, but it is clear even from the above short exposition that if indeed the energy spectrum is generated by a fractal domain composed of helical cells the local coupling hypothesis is wrong. The reduction of nonlinearity in helical cells inevitably depends on the coupling between even the smallest dissipation scales and the largest integral scale of turbulence.

The scaling analysis carried out in Levich (1980) and Levich (1987) showed that indeed the non-local in the Fourier space of wavenumbers interactions (or in the space of scales which are the inverted wavenumbers) is dominant. The non-locality of interactions does not totally destroy the local character of certain averaged quantities, such as the energy transfer in wave number space and this is what upholds the $-5/3$ energy spectrum in particular, but introduces another type of interaction, a non-dissipative "virtual" interaction between the nonlocal in Fourier space velocity harmonics. These non-local interactions although short time and in this sense virtual, nevertheless must be observable in particular in the energy spectrum for large enough values of wavenumbers close to the dissipation range. The theory predicted a hump in the energy spectrum over the Kolmogorov spectrum in near to dissipation part of the spectrum as a result of the non-local interactions. Such anomaly is well detected now in DNS, although only in the work of Mininni et.al. (2008a, 2008b) the possibility of connection between the energy spectrum excess over K41 and the helical structures was mentioned.

^f It may be of interest to analyze in retrospect the history of the helical concept. It was first brought forward by Levich and co-authors

in 1982 and 1983 publications and in different interpretation by Moffatt in 1985. With some exceptions the attitude of the professional community was showing little interest. To start with the existence and/or relevance of helicity associated effects in developed turbulence, e.g., Rogers and Moin (1987), Polifke (1991), Speziale (1987) was denied. The confusion was along two different lines. The anomalous alignment of \mathbf{v} and $\boldsymbol{\omega}$ was seen as a small effect with only relatively few disjoint points in the flow showing the alignment. The second line of denial was based on confusing the average helicity itself and the fluctuations of helicity in small sub-domains of the flow or short lived. Even some recent publications asserted that while helicity fluctuations can play role in the early stage of turbulence decay from certain initial conditions this role disappears when turbulence reaches the developed stage, e. g., Holm and Kerr (2007). Kerr and Holm correctly concluded that the early Euler stage of turbulence formation is accompanied by acute Beltramization, but concluded that later this Beltramization vanishes and failed to see the its influence on the developed turbulence dynamics, except of formation of vortex tubes. Apparently Holm and Kerr were unaware of a much earlier paper in which for the first time the acute, dramatic alignment of \mathbf{v} and $\boldsymbol{\omega}$ at the very initial stage of turbulence development was first demonstrated for the decaying Taylor-Green vortex (Shtilman, et.al., 1985). The Taylor-Green vortex imposes certain symmetries on the flow that allowed in this paper to carry out DNS with higher and respectable value of the Reynolds number, such that could have not been achieved at that time with no such symmetries imposed. The DNS showed in particular that the initial stage of Taylor-Green vortex decay goes through extreme Beltramization before moderating to the level that since then was observed in many papers on the subject. Interestingly the Taylor-Green vortex symmetries are widely used even now for achieving DNS with higher values of the Reynolds number. Since no helical structures were actually explicitly observed as decisively as they have been since then, the DNS capabilities were not sufficient for the purpose, the rebuttal of criticism was not easy. It is clear that the anomalous alignment effects are secondary in a way and without the whole concept of helical fluctuations as it was formulated above this effect cannot be dynamically important. Indeed, what dynamical effect can be played by 1% or 2% of disjoint

locations in the flow where the alignment takes place as was correctly argued in Rogers and Moin (1987) over twenty years ago. It is only when it is recognized that these are not disjoint locations but a cluster of helical structures, or a relict residue of short lived hierarchy of helical fluctuations that each contributed to the formation of this cluster then the importance of the effect can be gleaned. A good example can be the famous relict radiation with temperature $2.8K$ above absolute zero. What would be its significance for Universe evolution if it was seen in isolation rather than interpreted as a relict remainder of the Big Bang? The second issue is still plaguing the understanding of the phenomenon. There had been prior multitude of papers that correctly showed that helicity itself plays only a limited role in slowing down the energy cascade to small scales and to do this it should have the maximal possible value. But from general mathematical considerations, just the consequence of definitions of energy and helicity spectra and taking into account the well known from the Fourier analysis Schwartz inequality, puts upper bound on the helicity spectrum for a given energy spectrum. If the dynamics is also taken into account then this maximal value is even lower at small scales or the high wavenumbers whatever the helicity source at the large scales is. Since it is these harmonics that are most contributing to the energy dissipation and all other relevant turbulent fields such as enstrophy, etc., the usual wisdom is that helicity cannot play big role for the turbulence dynamics. The average helicity in fact can be shown to behave similarly to a passive scalar convected by the velocity field. But as soon as one invokes the concept of helicity fluctuations instead of the average helicity all this reasoning fails. The helical fluctuations are short lived in a mathematically well defined way and have the opposite signs of helicity. As a result the averaging over a time span much bigger than their own life time just eliminates them, they become effectively invisible. This is why they were called "virtual" in Levich (1987). Or the structures occupy a small volume domain and have the opposite sign helicity and hence screening each other. Then the averaging over space again eliminates them. But at the same time the helicity amplitudes inside these structures or fluctuations can be truly large, either for a short time span or in small sub-domains in space. In this case of course these fluctuations can influence dynamics by reducing the nonlinear coupling term in

the Navier-Stokes equations in the limit of high wavenumbers. From the statistical viewpoint the relevant measure of helicity fluctuations would be rather their correlations in physical and Fourier space. If the correlations were weak as they would be if the helicity fluctuations were near to the Gaussian distribution (Bell curve) this would indicate that they are not dynamically significant. But in reality they turn out to be acutely anomalous. It was shown theoretically that the Gaussian law for the helicity fluctuations contradicts to the assumptions of K41 and interferes with the energy cascade to high wavenumber harmonics and subsequent energy dissipation. For the cascade existence it is necessary that the fluctuations should be non-Gaussian and strongly correlated in the high wavenumber region. But on the other hand if they are strongly non-Gaussian this is cocommittant to intermittency. In fact the helicity fluctuations are responsible for intermittency. If it was not for the helicity fluctuations there would be no turbulence as we know it at all (Levich and Shtilman, 1988; Levich, et.al., 1991). The logical conclusion would be that the intermittent regions are not structureless as they were previously seen, but in fact they have well defined inner coherent flow structure. The helicity effects are truly subtle and difficult to reconcile with the usual statistical K41 phenomenological description of turbulence. The statistical description is limited if the most important regions of turbulent flows, actually the turbulent flow core consists of totally coherent Beltrami flow cells. It is complicated in the first place to define statistically the observable effects stemming from the presence of multiscale helicity fluctuations. Helicity structures in their stable form have mathematical sense when they are delineated from each other and from the surrounding flow by vorticity surfaces. This life time is short and the boundaries are impossible to observe in laboratory experiment where the measurements are all made at a point or a few points in the flow (see the next footnote). The only way to do it is visualization (Tsinober and Levich, 1983). Numerically this is also not easy and requires the Reynolds numbers at least as high as they have been achieved in modern DNS. The relict effect would be the ubiquitous alignment effect but it is not full proof effect and can be contested by skeptics. In Levich (1987) and in the previous works it was stated that the nonlinear interaction inside the helical fluctuations is reduced. To test this claim some tried to conditionally

sample the flow and look for the regions of low and high turbulent activity using several crude criteria for this sampling that were not always locally directly related to the nonlinear coupling term in the Navier-Stokes equation. The point is that although the non-linear coupling is reduced in helicity fluctuations but this is true only in a relative local sense in space and time. Concurrently it is this relative reduction that at the same time creates the relatively small size sub-domains with high amplitudes of turbulent activity. In other words the nonlinear coupling is reduced in the whole hierarchy of helical fluctuations everywhere inside the fluctuations during their life time by comparison with what it would have been for the same absolute values $|\mathbf{v}(\mathbf{r}, t)|$ and $|\boldsymbol{\omega}(\mathbf{r}, t)|$ if there was no alignment between the two vector fields. The global effect is thus impossible to observe just considering the regions of small and intense turbulent activity. Unless by extraction of the helical structures residing in the regions of intense turbulent activity and actually making sure that in these regions the alignment is acute. In the past there was only one attempt in Polifke and Levich (1990) to estimate the correlation between the extent of global alignment inside a helical fluctuation and the intensity of turbulent activity, but the results though indicative were far from convincing because the Reynolds number of DNS was by far too small for the intended conditional sampling procedure. A classical example of confusion in understanding this subtle effect of the nonlinear coupling reduction was the DNS of Kraichnan and Panda (1988). They found that the mean square nonlinear term in the Big-Box turbulence is somewhat smaller than it would have been for the Gaussian statistics of \mathbf{v} and $\boldsymbol{\omega}$ and then showed that the same happens for other dynamical systems and in particular such that have no dynamical intermittency. The result they obtained was really trivial and the reduction of this sort has nothing to do with the reduction of nonlinear coupling by the "cellular Beltramization" advanced by the helical concept. The genuine nonlinear coupling reduction takes place hierarchically with ultimate formation of the final sub-domain of high turbulent activity and in this sense the effect is as virtual as the hierarchy of helicity fluctuations themselves; to be sure a difficult concept for classical physics. It remains visible only inside the final stable sub-domain and is observed through the actual strong alignment. The alignment is the only way that it can be observed and

defined. Statistical comparisons have very limited applicability if the flow is made of quasi-Beltrami flows of opposite sign and fluctuating on a very fast time scale. The effects are almost invisible in crude statistically averaged quantities. The real progress in experimental validation, since DNS is in fact a crucial experimental proof of the existence of helical fluctuations in the sense described here has been achieved only recently with the direct visualization of helical structures (Mininni, et. al., 2008 a,b) . Let us consider now with a new level of comprehension the Figs. 2 and 3. The Fig. 2 shows typical coherent bands (or patches, or filaments) of intense vortical motion that is obviously organized. Although they occupy relatively small volume flow sub-domain they stretch through the whole flow domain in one dimension. In Fig. 3 the same flow sub-domain is shown but with local strong alignment of velocity and vorticity structures or cells indicated in red and blue. Each of these cells is like the one shown in Fig. 1. It is now obvious that the vortical bands consist of the helical cells of different sizes in a pattern that is obviously non-random in the distribution of the cells with the opposite sign of helicity. These "Beltrami and anti-Beltrami" tubular cells, as shown in Fig. 1, screen each other so that the total helicity remains small or zero. It is suggested naming the vortical bands in Fig. 2 and 3 Beltrami Cellular Clusters-**BCC**. The cells are tubular shape quasi-Beltrami flows of opposite sign of helicity clumping together and forming the clusters that are seen as stretched vorticity bands. The definition will help to fix the semantics of the phenomenon and avoid the repetition of using intermittently the words vortical patches, filaments and bands that the literature on turbulence is full of. And the conclusion is that all CS in turbulence are just aspects of **BCC**. It is instructive to delineate here between the concept as advanced in this paper and the previous works of Levich and co-authors, and the hypothesis expressed by Moffatt (1985) on the other. It was correctly pointed out in this latter work that Beltrami flows play a singled out role among all other Euler flows. However the conjecture that turbulence is made of Beltrami flows and the energy dissipation and other intense processes occur at the boundaries between them is not really correct as we know now. As was explained above the Beltrami flows and the boundaries between them are virtual, in the sense that they are shortlived: both the volume of the helical fluctuations and their life

time tend to zero when $Re \rightarrow \infty$. They stabilize only when both are squeezed into a diminishing sub-domain tending to fractal. Moffatt correctly understood that the Beltrami flows are singled out among other Euler steady flows and largely unstable. The truth is that with shortage of experimental data and insufficiency of numerical data it was somewhat difficult to put together the helicity fluctuations, fractals and the observed globally weak effect of alignment between the velocity and vorticity fields together in a consistent scenario. Still in the starting works of Levich and coauthors the hierarchy of helicity fluctuations forming a fractal and the consequent relative reduction of nonlinear coupling were clearly formulated but not directly related to Beltrami class of flows. Moffatt on the other hand correctly understood that the Beltrami flows are singled out among other Euler steady flows to be suitable candidate for the helical fluctuations. Historically both approaches should be seen in conjunction and as complementary to each other despite certain understandable mistakes and inaccuracies made at the time. Despite many years of disrepute interest of the professional community to helicity in turbulence has not died in the last 25 years or so. The persistence of the effect of alignment between velocity and vorticity that would be revealed to anyone doing DNS of turbulence flows, amplified by a lack of understanding of this stubborn effect would cause a good deal of denial among some. Still others tried to approach this ubiquitous feature in a more positive way. For instance Farge, et.al, (2001) correctly associated the alignment effect with coherence in wavelets analysis. However, the understanding that the alignment of \mathbf{v} and $\boldsymbol{\omega}$ is a relict effect of the hierarchical helicity fluctuations evolution culminating in the formation of **BCC** has been elusive. As was noted, although the alignment indicates the existence of helical structures but strictly speaking does not prove them. But all the attempts of alternative explanation did not succeed. Still only when the helical structures are identified as clearly as they are in Figs. 1 and 2, then their presence is uncontestable. One should compliment Mininni, et.al, (2008a and 2008b) for their clear testing and analysis of the old predictions and shedding new light on them.

Some eminent authors in geophysical studies recognized the intrinsically helical nature of atmospheric turbulence. For instance Mary Selvam (1988) wrote: "The atmospheric circulation patterns

therefore have fractal dimensions on all scales ranging from the planetary to the turbulence scale, the strikingly visible pattern of fractal geometry being exhibited by the clouds. The above concept of the steady state turbulent atmospheric boundary layer as a hierarchy of intrinsic helical fluctuations is in agreement with the theoretical investigations of hydrodynamic turbulence by Levich (1987). All basic mesoscale structures (less than 1000km in the tropics) appear to be distinctly helical. These include such outstanding examples of organized geophysical motion as medium scale tornado generating storms, squall lines, hurricanes, etc.” Another example is the classical works of Lilly on cumulus storms (1985 and 1986). But the concept still remained on the fringes of general research. Nevertheless, recent data leaves little doubt that indeed extreme helicity is present in many seemingly disparate geophysical events, from midlatitude storms to tropical cyclones, e. g., Molinari and Vollaro (2008). Such helical structure of geophysical events was explicitly predicted in Levich and Tzvetkov (1984, 1985). As was noted an unfortunate confusion of many papers of the last 25 years, including some cited above, is that they considered the global or statistically averaged helicity. I would like to reiterate that mean helicity by itself plays limited role in turbulence dynamics. It is the fluctuations of helicity of opposite sign that are all important. The global or mean helicity can stay zero or small. As was shown the restrictions of very general mathematical nature generally do not allow the global or average helicity to be large in turbulent flows in a dynamically meaningful way. But helicity in fluctuations can be maximal and in some it is. Then helicity is immensely significant dynamically. Just for the simple reason that it reduces the nonlinear coupling.

⁹ In laboratories the closest to homogeneous isotropic turbulence is created in the so-called flows past a grid. A flow of air or water (sometimes electrolyte, see below) or air goes past a grid and enters into a long tank with diverging walls, so that the typical transverse size is much greater than the grid size. The grid serves as a source triggering instabilities in the flow on the scale of the mesh size. Thus naturally the flow becomes turbulent and remains turbulent at some distance from the grid. The turbulent flow propagates in the vessel but since the integral scale of turbulent flow, comparable with the mesh scale is much smaller than the distance from the walls the lat-

ter red by the mesh is lost by the flow but turbulence amplitudes are still large. All these conditions can be and are properly defined mathematically by experimentalists. Such is the best approximation in laboratory conditions to the decaying homogeneous and isotropic turbulence model considered by Kolmogorov. The measurements are usually carried out by means of hot wire anemometers keenly responding to fluctuations in the fluid speed by corresponding fluctuations in temperature, which are subsequently converted to variations of the electrical current. Such hot wire anemometers are placed at one or at a few locations inside the turbulent flows and provide the time signals of velocity field variations in the flow. The simultaneous data from two or more local sensors allows, in principle, to compute the velocity field spatial derivatives as well. The accuracy of these measurements is naturally restricted by the relative size of the sensors and distance between them. Thus on the whole measurements furnish experimentalists with relatively scarce information as far as the global structure of turbulence is concerned. As a result experimentalists often take recourse in visualization techniques, e.g., injection of hydrogen bubbles in the flow, supplementary to measurements (Monin and Yaglom, 1975).

This visualization, however, is not always reliable, and often is not a quantitative method of a scientific study. In numerical simulations a decaying homogeneous isotropic turbulence is generated on a $3D$ lattice, e.g., a cubic box with periodic boundary conditions called BigBox turbulence. Turbulence starts from either some unstable laminar flow like Taylor-Green vortex or randomly chosen chaotic initial flow conditions. The flow then evolves in accordance with the Navier-Stokes equations that are calculated in small steps forward in time. After some time elapses the flow becomes turbulent. Since there is no source feeding energy into the flow eventually turbulence would decay totally due to viscosity similarly to turbulence past the grid large enough distance from the grid. The distance from the grid in laboratory experiments plays the role of time in the BigBox turbulence. But for intermediate times turbulence is well developed and can be regarded as quasi steady state since the typical turbulent times scales are small by comparison to the typical time during which the total energy of the flow would dissipate into heat. One also can simulate BigBox steady state homogeneous isotropic turbulence by introduc-

ing a stirring force, random or not acting largely at the large scales, usually comparable or somewhat less than the box size. Turbulence then develops at the scales much smaller than the stirring force scales. In the past doubts were cast on numerical simulations of this sort, in particular over the choice of periodic boundary conditions. With time it has become clear that the simulations furnish reliable data, at least for the flows with simple geometry, provided that the Reynolds numbers of corresponding simulations are high enough and the simulations are well resolved (that is the grid in space is dense enough, the time step of simulations is small enough and the number of grid points in space/time is big enough for the choice of the Reynolds number). However it is still not possible to simulate the flows with very high Reynolds numbers and complicated flow geometry. This will likely remain a problem for the foreseeable future. This is another reason why it is so important to have models of macro scale turbulent flows based on correct fundamental physics of micro scales of turbulence, these latter demanding most of the computational power. But to do this one must understand in the first place this correct physics to build correct models.

^h Generally the solutions of the Navier-Stokes equations describing smooth laminar flows are unstable in the limit of large Reynolds numbers and become turbulent. There are exceptions, like the round pipe flows, or a flow on a rotating disc. But in practice they also of course destabilize and become turbulent eventually, e.g., due to finite amplitude perturbations and in practice shape defects.

ⁱ The flows of incompressible inviscid fluids are usually interpreted as *Diffeomorphisms of differentiable invertible mappings in infinite-dimensional* configuration space of all fluid particles into themselves induced by their motion. However the mappings are generally not analytical in that they are not necessarily invertible. In simple language the Euler equations allow non-analytical solutions with generally infinite number of surfaces, or sheets at which the velocity field is tangentially discontinuous; fluid layers slide along each other. Such sheets necessarily develop for instance for the flows at the solid boundaries. The principles of aeronautics were based on this remarkable property. Since viscosity is small but not zero and instead the Navier-Stokes equations govern the flow the sheets of tangential discontinuities instead become the thin layers of intensive vorticity, the

so-called vortex sheets in turbulent flows.

^j Additionally in a compact fluid domain the magnetic lines will fold and if the fluid motion lacks reflectional symmetry, for instance possessing non-zero helicity they will be twisted. The stretching, twisting and folding of magnetic field lines is the elegant mechanism of exponential growth of the seed magnetic lines by the turbulent dynamo (e.g., Ruzmaikin and Zeldovich, 1983). The dynamo effect although appears at the first glance as a linear effect in reality is not because of the random nature of the velocity field. When the magnetic field is big enough in amplitude the equations of motion become properly non-linear. The exponential growth goes as long as it takes the magnetic field to start influencing the fluid motion that generates its growth. The vortex sheets are unstable and have the irresistible tendency to fold. But as it is known now while folding the sheets also foliates into **BCC** with subsequent quenching of the nonlinear interactions.

^k If analyticity of the velocity field is assumed then the vortex lines should lie on surfaces of tori only. But analyticity is not generally possible for the ideal dynamical evolution of frozen-in fields. If the analyticity assumption is relaxed then the surfaces can be arbitrarily complex as was explained by Moffatt (1985). The Euler equations for ideal fluids allow infinite number of solutions with tangential discontinuities of velocity and/or its derivatives. Such surfaces can be seen as vorticity sheets and they are typical for ideal fluid dynamics and are widely considered in aeronautics in particular as approximation to viscous flow dynamics. For viscous fluids with vanishingly small viscosity $\nu \rightarrow 0$, or what is the same $Re \rightarrow \infty$, the velocity is of course analytical everywhere, but its space derivatives may become arbitrary large in a surface like domains in this limit. The domains will be strongly $2D$ with thickness tending to zero together with viscosity. Such surface like domains of large vorticity would be the equivalent of the vortex sheets in ideal flows.

^l The $-5/3$ energy spectrum was actually formulated in its modern form by Obukhov after the work of Kolmogorov of 1941 which formulated and axiomized the main scaling principles for HIT. Many years of experimental studies of clouds and analysis of atmospheric turbulent events in general and in particular associated phenomena, rainfall, pollutants distribution in the atmosphere, etc., carried out

for over 25 years by Lovejoy and Schertzer (e., g., Lovejoy, et al., 2008) point out that the Kolmogorov-Obukhov energy spectrum in horizontal plane holds approximately for orders of scales in atmospheric turbulence. In fact the law holds on the scales much larger than would be possible to regard the atmospheric turbulence as isotropic, if there are such isotropic scales at all. The height of the atmosphere is restricted to about 10 – 15 *km* and for larger horizontal scales the motions are obviously not isotropic. The atmosphere is stratified and greatly affected by gravity and buoyancy. Nevertheless, the $-5/3$ spectrum holds approximately as a function of horizontal scale up to at least to mesoscale (and may be further to planetary scales) and seemingly all the way down to millimeters. Despite experimental uncertainties K41 is one of the most stupendous scaling laws of nature. But of course only in conjunction with intermittency and most likely fractality of atmospheric turbulence. The puzzle to resolve is why despite the **BCC** and fractal composition of atmospheric turbulence the K41 spectrum itself is still there.

^m This is exactly the situation in atmospheric turbulence advanced and analyzed in the works of Lovejoy, Schertzer and co-workers. From the very early works they advance the point that atmospheric turbulence is neither $3D$ nor $2D$. The classical school of geophysics for many years were treating atmosphere as either $3D$ at relatively small scales not exceeding in horizontal its vertical height about $L_z \approx 10 \text{ km}$, or $2D$ at the scales exceeding this one, $L_{x,y} \geq L_z$. This implies that at about $L_z \approx 10 \text{ km}$ the energy spectrum should show so-called mesoscale gap, going from K41 appropriate for $3D$ turbulence to k_{-3} energy spectrum predicted for $2D$ turbulence. Also intuitively it may seem that at very large horizontal scales the atmosphere is totally flat. This is not true at all and the atmosphere retains its three-dimensional nature at any every horizontal scale. Observations rule out $2D$ atmospheric turbulence and mesoscale energy spectrum gap. Instead turbulent structures scale differently under scaling transformations and for a very "flat" situation the dimensionality of the active part of turbulence ostensibly becomes more and more stratified as is described by (5.9).

ⁿ As is clear the fractal and multifractal models described so far is a pure phenomenology that has little contact with the Navier-Stokes equations. Nevertheless, it is a very important phenomenology that

sheds much light on the real structure of turbulence and atmospheric phenomena. Multifractals appear always when the increases of intensity of fluctuations from the mean of a physical field are confined to sub-domains of progressively lower dimension and at the same time the increase of the intensity of fluctuations is ruled by a probability law with algebraic tail. DNS is still not and will not be able to establish the multifractal structure with certainty for the foreseeable future. Therefore only the analysis of geophysical data should be relied upon.

° There were many closure schemes developed in the last 70 years. The closures in one way or another creep into all contemporary meteorological and geophysical models. The reason for this is the same. There are no other dynamical approaches and descriptions of turbulence. Especially dangerous is the fact that the modeling of high wavenumber turbulence is usually done by closures or equivalent assumptions. The high wavenumber turbulence is all coherent and such assumptions are definitely wrong. One should be puzzled however that in many cases the models work well enough for applications, but not at all always. It is important to understand when and why they do or do not.

The Boltzman equations come up in many situations when strongly interacting Hamiltonian dynamical systems are treated in so-called random phase approximation-RPA. Essentially the RPA assumes that the relevant fields are incoherent and their phases are quasi-random. Quasi-random because if they were totally random than there would be no nonlinear coupling at all and no Boltzman equation. But the closures are implemented in such a way that all orders of the correlation functions are factorized as powers of the lowest order correlation function. These approximations became a direction of research as "weak" plasma turbulence. The spectra in these problems are always Kolmogorov like power laws (but not necessarily $-5/3$ power law) that are the solutions of the Boltzman equation for particular problems of interacting plasma waves with dispersion (e.g., Zakharov, Works on General Theory of Waves and Turbulence in Nonlinear Media, website; Zakharov and Kuznetsov, 1997). The weak turbulence theories for nonlinear waves are somewhat better justified than K41 theory because the waves have dispersion relation and therefore some well defined linear approximation that does not exist in fluid

mechanical turbulence. Nevertheless, I believe that weak turbulence approximation rarely realizes if at all. The coherence and intrinsic necessity of organization are likely to be general laws for most if not all dissipative dynamical systems with many degrees of freedom.

^p There is no doubt that there are deep reasons for this. If it was not like this no engineering treatment of turbulence would have been possible. Much of the large scale mean properties of turbulent flows are well modeled using the assumption that the small scale turbulence is essentially the Kolmogorov like turbulence with no regard of CS or intermittency. And this fact underlines much of the modeling of atmospheric turbulent flows as well. Because of this when one appeals to obvious observation of CS the usual answer of the proponents of the models would be that this all may be true about CS but their origin is in boundary layer effects and has relevance only for details of turbulent motion near boundaries, while the small scale turbulence away from the boundaries is properly described by K41 theory. As far as intermittency of small scale turbulence is concerned the reaction of many would be that it is the effect of high order velocity correlation functions not affecting mean large scale flow properties. As was commented before only in geophysical turbulent flows where we can watch turbulence over many orders of magnitudes and see CS and their manifestations with naked eye that one starts believing that K41 theory is acutely insufficient at all scales. But even here some would argue that this is because of gravitation and stratification of geophysical flows so that no pure HIT model can be realized. As far as violent events are concerned they are investigated as separate phenomena, one by one and often don't seen as manifestations of global turbulence. The contrary arguments will continue falling flat on many ears unless a new theory would be compatible with the models of turbulence based on pure or somewhat patched up K41 theory for the small scale turbulence and at the same time account for the CS and intermittency. In other words a new theory must be complementary rather than obstructive. From this viewpoint it is possible to understand the excitement caused by RNG methods among many researchers some 20 years ago. At the same time among others concerned with fundamental aspects of turbulence the same RNG fever often caused only irritation and rejection. It is indeed that fundamentally RNG did not explain anything on the nature of turbulence

beyond the many previous closures and perturbation theories. But RNG allowed to construct models of BL flows in a certain unified and systematic way and created an appearance of a theory (this practical approach was first developed and advanced in a series of papers written by Yakhot and Orszag (1986) and references therein).

⁹ Such convection was discussed in the previous section in a context of the usual perturbation theories. The idea was that mere convection by large eddies of the small ones should not have direct dynamical significance and can be removed in all the orders of the perturbation expansion by the Galilean transformation $f \rightarrow \pm \mathbf{k} \cdot \mathbf{u}$, but applied for random convections of small eddies by the ensemble of the large ones. This is why it is called "random Galilean transformation". But of course this does not happen for the following reasons. If this was true then only the interactions between the close in Fourier space triads of velocity harmonics (7.29), or eddies of comparable size, would be interacting to each other in a dynamically significant manner. In this case as was explained above the only solutions that any perturbation theory can result in would be the K41 spectrum, but also the analogous K41 scaling relations for the higher order velocity field statistics. In other words the original K41 theory would be the only possible one, with no intermittency and no CS.

The above line of reasoning had had definite historical merits for understanding the principles of locality of energy transfer in *k-space*, which is likely to be correct in an asymptotic way, but surely cannot be seen as proof of anything. It just tells one that the perturbation theories per se are not capable to describe the singular fractal sub-domain such as **BCC**. Indeed, how is it possible for instance to perturbatively approach the Beltrami flow from a random flow with no structure? On the contrary with the assumptions made above while formulating the model Eqs. (8.30) we are trying to solve a much lesser in complexity problem. Reiterating we are trying to determine the fractal dimension of conjectured flow sub-domain dominated by high wave number harmonics, the ultraviolet asymptotic limit, and confined in physical space to a fractal sub-domain that creates around it a quasi-Gaussian flow with K41 spectrum and much less intensive high wavenumber content. Not to be confused any more with the possibility of subtracting the highest order nonlinear terms generated by the perturbation expansion it is reminded that in fact the nonlinear

coupling in the model Eqs. (8.30) has the vertex $\eta_{ijs} = P_{ijs}(1 + \alpha)$ and not P_{ijs} itself. This was disregarded for simplicity because the scaling result (8.48) would remain the same with substitution of \mathbf{k} by $\mathbf{k}(1 + \alpha)$ with $\alpha = \alpha(\mathbf{k}, f)$ from (8.32).

^r Implicit in the theories of turbulence based on perturbation expansions that were analyzed in Section 7 was that in order to obtain the K41 spectrum as a solution it would be necessary to find a suitable mechanism for cancellation of all these singular terms in powers of $L^{2/3}$. Singular in the sense that the limit $L \rightarrow \infty$ is equivalent to the limit $Re \rightarrow \infty$. One of the assumptions was the "random" Galilean invariance discussed in the previous footnote. If this really happened than in the terms of reduction of the nonlinear coupling this would have meant a stronger reduction with effective coupling constant $\lambda(k)^{Kraichman} \propto k^{-1/3}$, instead of $\lambda(k) \propto k^{-1/6}$, as in (8.60). This is what in fact the idea of random Galilean transformation as a tool for the regularization of the perturbation expansions is all about.

^s I would like to mention that following Radkevitch, et. al. (2007) and Radkevitch, et. al. (2008) it is particularly suitable to consider **BCC** in terms of $(3 + 1)$ space-time dynamics. The notion of virtual helicity fluctuations can be then interpreted as occupying a certain space-time domain that is $D_{st} = (3 - \mu) + 2/3 + \mu/3 = 10/3$. This is equivalent to the total reduction of the nonlinear coupling by $(l/L)^{-4\mu/3} \rightarrow Re^{-\mu} = Re^{-1/2}$. Although, it may seem somewhat superficial to use this terminology outside of quantum physics it may be useful nevertheless, since the terminology of virtual helicity fluctuations was introduced, to invoke a related "uncertainty" principle here. The helicity fluctuations may be spatially filling with their life time tending to zero in accordance with the value of D_{st} , or they live in the volume fraction tending to zero but their life time is finite, or any intermediate situation in such a way that that the total $(3 + 1)$ volume has the dimension $D_{st} = 10/3$. The mathematical procedure applied here cannot distinguish between the respective realizations of turbulence since their contributions are of the same order. However the energy spectra corresponding to these realizations are not the K41 anymore. Let us consider briefly how it works. Let us write the expression for $(3 + 1)$ - *space/time* dimensionless volume element corresponding to the above D as follows:

$$\begin{aligned} \delta V^{3+1} &= \delta V_B(l)\delta T_B(l)/V(L_i)\Delta t^{K41}(l) = \\ &= \delta V_B(l)\delta T_B(l)/L_i^3 < \epsilon >^{-1/3} l^{2/3} = (l/L_i)^{2/3} = Re(l)^{-1/2}, \quad (S.1) \end{aligned}$$

where $\Delta t^{K41}(l) \approx < \epsilon >^{-1/3} l^{2/3}$ and both intervals are from the appropriate inertial ranges $L \gg L_i \gg l \gg l_d^n$, generally different for different fractal dimensions $D_F^n < D$. As long as **BCC** δV^{3+1} volume satisfies the relation (S.1) the corresponding contributions in the functional integral will be the same order of magnitude in the scaling sense. For instance consider **BCC** having life time $\delta T_B(l) = \Delta^{K41}(l)$. From (S.1) we obtain that the volume fraction:

$$\delta V_B(l)/V(L_I) = \{(l/L_I)^{2/3}/\delta T_B(l)\}\{\Delta t^{K41}\} = (l/L_i)^{2/3} = Re(l)^{-1/2} \quad (S.2)$$

and the corresponding fractal dimension is now $D_I^F = 7/3 < D_F$. But D_F^l should be also be an adequate solution of the functional integral representation in the sense that the corresponding perturbation expansion is (marginally) convergent. In other words the relation (8.58) must be fulfilled. Substituting the dynamical exponent $z^l = z^{K41} = 2/3$ we get $y^l = D + 2/3$, and the corresponding energy spectrum exponent $[E(k)] = (D - 1) - y^l - z^l = -7/3$. We notice now that $[E(k)] - [E^l(k)] = -2/3$, and in consequence, in the ultraviolet limit $(l/L_i) \rightarrow 0$, the contribution of $E^l(k^{-1})/E(k = l^{-1}) \propto (l/L_i)^{2/3} \rightarrow 0$. The same will be true for any other **BCC** with the life time bigger than the one determined for the leading **BCC** with $D_F = 5/2$, this latter generating the K41 energy spectrum. Thus there is a family of solutions, in feat ralizations of turbulence, with increasingly longer life times **BCC** corresponding to a multifractal structure of sub-domains with decreasing $D_F^n \leq D_F = 2.5$. Although the contribution of these sub-domains into the energy spectrum is vanishing in the ultraviolet limit the situation with the higher order turbulent field statistics can be the opposite, with the dominant contribution coming from the successively smaller sets and accordingly the higher wavenumbers in Fourier space, $k_d^n > k_d^{n-1} > k_d$. There are also solutions with $D_F^n \geq D_F = 2.5$ compatible with (8.58) and corresponding **BCC** life times smaller than for the leading sub-domain. For such realizations the energy spectra are on the contrary flatter than the K41 spectrum. But for these realizations $k_d^n < k_d^{n-1} < k'_d$ and they are effectively eliminated out by the viscous dissipation on time scales tending to

zero by comparison with the K41 eddy turnover time. Therefore an important conclusion follows that the K41 spectrum is singled out in the sense that although all other two parametric turbulence realizations corresponding to different values of (y, z) parameters contribute comparably to the generating functional, nevertheless in conjunction with the viscous dissipation the most observable spectrum is K41. On the other hand the **BCC** set responsible for the K41 spectrum are not most longlived. The ones who survive longer and most likely to be observed as stable structures seem to be supported by smaller dimensions domains. In particular if we consider what seems to be the longest life time **BCC** with $z \rightarrow 0$ then the corresponding minimal $D_F^{min} \rightarrow 5/3$. It should be noted that the amplitudes of turbulent fields in the fractal sub-domains can be very high, since their integral contributions to the functional integral are of the same order as from the averaged flow in the whole $3D$ domain despite the vanishingly small volume that they occupy. This and the longevity of these subdomains can be particularly important in anisotropic and BL turbulence. It appears that the model considered in this paper gives a glimpse into the multifractal structure of turbulence corresponding to the dynamical hierarchy of helical fluctuations. This problem is extremely complicated and out of scope of this paper, except from the above quantitative observations.

^t One can see in Figs. 11 and 12 from Mininni, et. al, (2008b) that the energy spectrum seems to lie between the trial spectra compensated respectively by the factors with exponents $5/3$ and $4/3$. The flatter than $-5/3$ slope of the part of the energy spectrum is sometimes referred to as the "bottleneck" problem that arises due to "depletion" of the nonlinear coupling. In fact there is no bottleneck problem and no undue depletion of the nonlinear coupling. But there is a normal and appropriate reduction of the nonlinear coupling due to the helicity fluctuations and **BCC**. The energy spectrum as a result is as it should be in both the inertial range and the *buffer zone*. The experimental and numerical data usually quote slightly lesser values of the skewness anomalous exponent. However the experiment is not reliable and DNS are still dealing with relatively low values of Re . The numbers are small enough to be compatible given the lack of accuracy and low Re . Since skewness is anomalously divergent when compared with this means that $\nu \int k^4 E(k) dk$ is also

anomalously divergent. This is not surprising since we established that $E(k)$ has excess over K41 spectrum in the buffer zone. However this should be at the same time that $\nu \int k^2 E(k) dk = \langle \epsilon \rangle$ is Re independent. The latter expression is a definition that attests to the fact of energy dissipation and indicates the constancy of energy flux in the space of scales. Analytically however the two conditions are difficult to reconcile. It was suggested in Levich (1987) that the way to do it is to assume small but non-analytical Re dependent correction to the velocity correlation function (3.14) with their origin in \mathbf{v}_{sing} . This non-analytical addition would be responsible for the excess of energy in the buffer zone in k -space, but is not noticeable in physical space because it tends to zero for large Re . But when differentiated sufficient number of times the small correction becomes large and dominant, as it happens for skewness in particular. I see no other way in fact to match K41 spectrum and (9.24). This cannot be done in a manner that is totally Re independent. Somewhere between the two spectra there must be a Re dependent transitional region. In this sense I suspect that in difference to K41 spectrum, even after stripping it from some of the glory typical for K41 theory, the (9.24) spectrum is not fundamental. The matching that was done here is not exactly unique since it is global in nature. For instance it can be assumed that spectrum in the *buffer zone* is still K41, but the constant in front of the power law is $\sim Re^{1/8}$ and the global contribution of the *buffer zone* into the path integral would be the same. As well as the result for skewness (9.27), etc. I would like to comment that Mininni, et. al, (2008b) cautiously assert that the boundary between the inertial range and the region of anomalous energy spectrum growth is Re independent and hence is likely a property of viscous range. In NT developed here the boundary is Reynolds dependent since its position is given by (9.3). However this dependence is so weak that it will require verification over a significant range of Reynolds numbers to disprove. As it is there is no real physical or observed fact that would establish beyond doubt the total Re independence of the location of the *buffer zone* in relation to the inertial range.

^u Note that the solution (8.64) does not satisfy the general scaling rules of the Navier-Stokes equations (7.1). Although, the scaling rules of the Euler part of the equations are of course met. What it means is that the solution does not match smoothly the viscous

dissipation term in the Navier-Stokes equations. This is a hint that non-analyticity is required, in the limit $Re \rightarrow \infty$, for the solution to match the inertial range with the *buffer zone*, as was discussed.

^v Using the spectrum (9.24) and still assuming the validity of (6.20) would lead to $U^+ \propto (y^+)^{1/16} Re_\tau u^{-1/16}$, a law that practically cannot be distinguished, with an appropriate choice of constants, from the logarithmic law in a wide enough range of scales and Re_τ . It is clear that the law can be true only in a very narrow region of the *buffer zone*. The nonlinear coupling due to the fluctuating velocity component and that due to the mean velocity field should be considered together and this will likely affect the structure of **BCC** in BL as compared the channel or pipe flow. In viscous sublayer $U^+ \sim y^+$, so that to match it at $y^+ \approx 5$. In the *buffer zone* U^+ should undergo quite an adjustment and in non-universal way of course. In any case the behavior of U^+ in the *buffer zone* seems as complicated as of the energy spectrum in the *buffer zone* in HIT. For practical purposes the profile in the buffer zone is usually modeled by trial functions to fit the experiment (e.g., B. Levich, 1962). But fundamentally the buffer zone remains very poorly understood.

But using the spectrum (9.24) in the *buffer zone* for trying to deduce the velocity profile may be not a good idea after all. The matter is that if indeed the *buffer zone* is the internal part of wall **BCC** the intensity of fluctuating velocities and their derivatives should considerably increase. Because extending the analogy with HIT these would correspond to \mathbf{v}_{sing} , a singular part of the velocity field and as such it should contribute comparably with \mathbf{v}_{reg} , a regular component, in the sense of nonlinear coupling in all channel flow domain. In HIT it was expressed by the scaling relations (8.24) and (8.25). I don't know how really to do it in inhomogeneous flow. But as a very rough estimate it can be assumed that,

$$\int_{\delta} RE_\tau^{1/2} \{ \langle \partial_{y^+}(u^+v^+) \rangle \}^2 >' dy^+ \approx \int \{ \langle \partial_{y^+}(u^+v^+) \rangle \}^2 >' dy^+, \quad (U.1)$$

where $\partial_{y^+}(u^+v^+)$ substitutes the nonlinear coupling (8.2). It is supposed to reflect a situation when the time averaged square of nonlinear coupling corresponding to **BCC** when averaged over the corresponding fractal volume $\propto Re_\tau$ would contribute equally with the nonlinear coupling averaged over the whole flow volume *propto* Re_τ , in

the sense similar to the one defined for HIT in Sections 8,9. However we automatically assume that the only dynamically significant velocity field component is the fluctuating one. This may be untrue. For instance in BL turbulence it would definitely be no true. The channel and pipe flows are degenerate so the nonlinear coupling involving the mean flow is zero. But in growing in thickness BL turbulence this does not happen of course. The nonlinear coupling due to the fluctuating velocity component and the nonlinear coupling due to the mean velocity field should be considered together and this will likely affect the structure of **BCC** in BL as compared with the channel or pipe flow.

^w The effect of phase decorrelations is equivalent to a forcing term in the considered together and this will likely affect the structure of **BCC** in BL as compared the channel or pipe flow. Navier-Stokes equations (1.1), $\mathbf{F} = \sum \mathbf{F}_n \{\mathbf{v}\} \delta(t - T_n)$, where T_n are the time intervals between the phase decorrelations. This can be done regularly or T_n can be also a shot-nose function. The force is a function of the flow itself and is workless, $\mathbf{F} \cdot \mathbf{v} = 0$. The forcing does not violate the incompressibility of course. At times intervals the selected Fourier coefficients are given a random shift. The velocity vector field in Fourier space consists of its imaginary and real parts and the angle between them that is chosen as a phase. It is enough to give random shifts at time intervals to these phases for the selected harmonics. For the details we refer to Handler, et. al. (1993) and Murakami, et. al. (1992).

^x The helical composition of turbulent jets was anticipated in Tsinober and Levich (1983). In the paper of Gavita, et.al., 2008, the authors claim that "contrary to the theories of Moffatt and Levich" they see intensive helicity in the regions associated with high turbulence activity and not the low one. It should be pointed out that indeed in several papers on the subject it was loosely formulated by some that helicity is high where the energy dissipation and likewise turbulent fields are weak. In reality of course this is a part of confusion with the concept of reduction of the nonlinear coupling due to the alignment and high helicity. The nonlinear coupling is indeed reduced in the presence of strong alignment. But this is a relative reduction that makes the coupling less than it would have been for the same absolute values of velocity and vorticity but in the absence

of alignment. But since this process takes place at virtual times at all scales as is explained in Section 5 and squeezes the active regions into a fractal asymptotically, the surviving **BCC** coincide with the regions of highest turbulent activity. And here the reducing role of alignment is clearly revealed as it results in the damping of the non-linear coupling constant. This is what has been usually meant in most publications, e.g., Levich (1987) and Polifke and Levich (1993). See also the previous Endnote f for more explicit explanation.

^y It is possible to imagine in principle that there are "silly" non-equilibrium mechanisms of energy conversion. For instance K41 theory assumes that while loosing its coherence energy injected into flow does not serve to anything useful, which is to say that no inner organization is created. The theories of weak turbulence describe similar processes when nothing interesting is created while energy flows to small scale motion and dissipated. However as I noted above such systems are based on approximations that are likely not to be true in real situations. Even if for some problems it is possible in a limited sense to assume weak interaction approximation the dynamics will drive itself the system out of the scope of this assumption and coherence effects will assert themselves.

Appendix A:

Topological properties of Euler flows.

Moffatt (1985) proved a very powerful theorem that surprisingly has to some extent remained underutilized by turbulence community; although in my view it has definite bearing on the structure of turbulence and is particularly relevant for **BCC** in turbulent flows.

By Euler flows one means stationary solutions of the Euler equations (1.6) that we write now in the vorticity form as follows:

$$\partial_t \omega - \text{curl}[\mathbf{v} \times \omega] = 0. \quad (A.1)$$

The stationary solutions of (A.1) are of the two types. One is:

$$[\mathbf{v}_E \times \omega_E] = \nabla \alpha, \quad (A.2)$$

where in particular:

$$\alpha = (P + \mathbf{v}^2/2) = P' \quad (A.3)$$

Forming a scalar product of (A.2) and \mathbf{v} and ω we obtain:

$$\nabla\alpha \cdot \mathbf{v}_E = \nabla\alpha \cdot \omega_E = 0. \quad (A.4)$$

Meaning that the streamlines and the vorticity lines lie on the surfaces $\alpha = \text{const}$, excluding the critical surfaces $\nabla\alpha = 0$.

The other type of solutions is:

$$\omega_E = \zeta(\mathbf{r})\mathbf{v}_E.$$

These are actually the Beltrami-Gromeko flows. From continuity $\nabla \cdot \omega_E = \nabla \cdot \mathbf{v}_E = 0$ we obtain:

$$\nabla\zeta \cdot \omega_E = \nabla \cdot \zeta\mathbf{v}_E = 0 \quad (A.5)$$

Meaning that the streamlines and the vorticity lines lie on the surfaces $\zeta(\mathbf{r}) = \text{const}$, i.e., they belong to the same class of solutions as the ones in (A.2). There remains a degenerate case of solutions:

$$\omega_{EE} = \zeta\mathbf{v}_{EE},$$

$$\zeta = \text{const}. \quad (A.6)$$

These solutions are singled out because the topology of the streamlines and the vorticity lines in this case is different. As was first explained by Arnold (1974) these are the only Euler flows with the volume filling ergodic streamlines and vorticity lines. This is why we chose a subscript EE-Euler ergodic. Since developed turbulence is almost certainly an ergodic phenomenon it is strongly felt that these solutions are likely to have direct relation to **BCC**.

Therefore the argument may be that they have no chance to be realized in turbulent flows. Note also that the Euler ergodic flows \mathbf{v}_{EE} have counterpart in the Navier-Stokes equations. Indeed, any flow such that:

$$\mathbf{v}_{NE} = \zeta\omega_{NE} = \mathbf{v}_{EE}\exp(-\nu\zeta^2t), \quad (A.7)$$

is the exact solution of the Navier-Stokes equations having in literature the name of Trkal solutions. Thus for any ω_{EE} flow with an arbitrary topology of (\mathbf{v}, ω) - *lines* there exists a corresponding ω_{NE} with equivalent topology of (\mathbf{v}, ω) . The use of the "equivalent" is not accidental, because the topologies of the two are not identical.

It was shown by Arnold that if \mathbf{v}_E is analytical then the only surfaces on which \mathbf{v}_E - *lines* can lie are tori with the \mathbf{v}_E - *lines* winding around them, closed or ergodically. The tori can be linked to each other in an arbitrary way. As far as the analytical $\mathbf{v}_E E$ are concerned they can exist only for very particular boundary conditions, e.g., periodic boundary (Arnold, 1974). Such limitations are the consequences of special properties of the $curl^{-1}$ operator that allow only very sparse set of eigen values. In other words the analytical Euler flows should have very special topology of \mathbf{v}_{EE} - *lines* and hence are few in the functional space of solutions of Euler equations and hence cannot be attractors. In this sense they are not interesting for turbulent flows since such particular restrictions on the streamlines and vorticity lines topology are totally unfeasible for the arbitrary complex motion that is associated with turbulence.

Moffatt made a break with these limitations by observing a rather obvious fact that the solutions of Euler (inviscid) equations allow an arbitrary number of discontinuous solutions, such as tangential discontinuities of the velocity field, the seats of vorticity sheets. Such vorticity sheets are fundamental for fluid mechanics. Their viscous analogue in the limit $Re \rightarrow \infty$ is behind much of the aeronautics and the lift force origin in particular. In other words there is no reason at all to limit the consideration of Euler flows with analytical fields. This seemingly simple reassessment of Arnold's classical results led Moffatt to most nontrivial and remarkable conclusions.

Here I will give a simplified exposition of these conclusions referring for details to the seminal paper of Moffatt (1985).

Moffatt Theorem: For an arbitrary smooth differentiable velocity field \mathbf{v}_0 of arbitrary topological complexity of streamlines in a given closed domain D in \mathbf{R}^3 , there exists at least one topologically accessible Euler flow \mathbf{v}_E .

Proof: It follows from the analogy between Euler flows \mathbf{v}_E and steady state flows in MHD, \mathbf{B}_M . The full MHD equations for a single

fluid are as follows:

$$\begin{aligned} \partial_t \mathbf{v} - [\mathbf{v} \times \boldsymbol{\omega}] &= -\nabla P' + [\text{curl} \mathbf{B} \times \mathbf{B}] + \nu \Delta \mathbf{v}, \\ \partial_t \mathbf{B} &= \text{curl}[\mathbf{v} \times \mathbf{B}] + \eta \Delta \mathbf{B}, \\ \nabla \cdot \mathbf{v} &= \nabla \cdot \mathbf{B} = 0. \end{aligned} \tag{A.8}$$

We assume now that $\eta = 0$, but at the same time $\nu \neq 0$ and finite. Hence the appropriate boundary conditions for the system (A.7) can be chosen:

$$\begin{aligned} \mathbf{v}|_{\partial D} &= 0, \\ \mathbf{B} \cdot \mathbf{n}|_{\partial D} &= 0 \end{aligned} \tag{A.9}$$

for all times $t \geq 0$. Since the first of the Eqs. (A.8) is purely dissipative we can be sure that as $t \rightarrow \infty$, the velocity field relaxes to $\mathbf{v}(t \rightarrow \infty) \rightarrow 0$. Since $\nu \neq 0$ and can be chosen arbitrarily big there is no danger of developing a singularity anywhere in $\mathbf{v}(\mathbf{r}, t)$. In a steady state:

$$[\text{curl} \mathbf{B}_M \times \mathbf{B}_M] = [\mathbf{j}_M \times \mathbf{B}_M] = \nabla P'. \tag{A.10}$$

We notice that by the following substitution:

$$\begin{aligned} \mathbf{j}_M &\rightarrow \boldsymbol{\omega}_M, \\ \mathbf{B}_M &\rightarrow \mathbf{v}_M, \\ P' &\rightarrow P', \end{aligned} \tag{A.11}$$

the Eq. (A.10) for \mathbf{B}_M becomes identical with Eq. (A.2) for \mathbf{v}_E . Hence for every solution \mathbf{B}_M there is an identical solution \mathbf{v}_E and the number of \mathbf{v}_E in the functional space of all Euler equations solutions equals to the number of \mathbf{B}_M solutions. Note that the one to one correspondence is between \mathbf{v}_E - *lines* and \mathbf{B}_M - *lines* rather than between $\boldsymbol{\omega}$ - *lines* and \mathbf{B}_M - *lines*. Apart from the cross helicity invariant defined in Section 2 (footnote 17), both are the exact invariants for $\nu = 0$, the MHD equations have another invariant for the conductivity $\eta = 0$, the magnetic helicity:

$$\begin{aligned} H^M &= \int_D \mathbf{A} \cdot \mathbf{B} dV, \\ \mathbf{B} &= \text{curl} \mathbf{A}. \end{aligned} \tag{A.12}$$

As is well known the magnetic helicity is a constraint on the relaxation of magnetic field:

$$\int_D \mathbf{B}^2 dV \geq M_{min} > 0. \quad (A.13)$$

Briefly the proof is like this. From the Schwartz inequality we have:

$$\int_D \mathbf{B}^2 dV \int_D \mathbf{A}^2 dV \geq \left\{ \int_D (\mathbf{A} \cdot \mathbf{B}) dV \right\}^2. \quad (A.14)$$

By variational principle it can be shown that:

$$\int_D \mathbf{B}^2 dV \geq q_0^2 \int_D \mathbf{A}^2 dV, \quad (A.15)$$

where q_0^2 is the square of the minimal eigen value for the solution of the diffusion eigen value problem:

$$\begin{aligned} (\Delta + q_0^2)\mathbf{A}(\mathbf{r} \in D) &= 0, \\ \text{curl}\mathbf{A}(\mathbf{r} \notin D) &= 0, \end{aligned} \quad (A.16)$$

where \mathbf{A} is continuous across ∂d and $|\mathbf{A}(|\mathbf{r}| \rightarrow \infty)| \rightarrow 0$. Comparing (A.15) and (A.14) we obtain:

$$\int_D \mathbf{B}^2 dV \geq q_0 |H^M| = M_{min} > 0. \quad (A.17)$$

Hence:

$$\int_D \{(\mathbf{B}(t \rightarrow \infty))^2 + \mathbf{v}(t \rightarrow \infty)^2\} dV \rightarrow M_{min}, \quad (A.18)$$

monotonically, since $\mathbf{v}(t \rightarrow \infty) \rightarrow 0$. Hence, for arbitrary complex topology of $\mathbf{B}(t \rightarrow \infty) = \mathbf{B}_M$, i.e., $\forall \mathbf{B}(t = 0) : \exists \mathbf{B}_M$, where \mathbf{B}_M - *lines* have equivalent topology that is *dynamically* accessible. This is not possible in the class of analytical functions as was explained above. The conclusion is that the \mathbf{B}_M generally are not analytical. For any $\mathbf{B}(t = 0)$ there may be generally many corresponding *dy-*
namically accessible \mathbf{B}_M , but it is enough to assume that there is at least one. Since there is an uncountable infinity of $\mathbf{B}(t = 0)$ there is also an uncountable infinity of \mathbf{B}_M that are *dynamically* accessible

and having equivalent topology, in the sense that in difference to the initially smooth $\mathbf{B}(t = 0)$ there should be a subdomain in D where \mathbf{B}_M , or its derivatives suffer tangential discontinuities.

Since $\forall \mathbf{B}_M : \exists \mathbf{v}_E$, we conclude that there is an uncountable infinite set of \mathbf{v}_E in the functional set of all solutions of Euler equations. In difference to \mathbf{B}_M that are dynamically accessible from the topologically equivalent $\mathbf{B}(t = 0)$, generally no such dynamical accessibility exists for \mathbf{v}_E , i.e., generally $\lim \mathbf{v}(t \rightarrow \infty) \neq \mathbf{v}_E$. Therefore the only statement that can be made is that $\forall \mathbf{v}(t = 0) : \exists \mathbf{v}_E$, with arbitrary complex topology of $\mathbf{v}(t = 0)$ - *lines*, that is *topologically* accessible, in contrast to *dynamically* accessible. Exactly the same applies to ergodic flow field realizations. For any $\mathbf{v}(t = 0)$ with ergodic volume filling topology $\mathbf{v}(t = 0)$ - *lines* there is at least one topologically accessible \mathbf{v}_{EE} , which means that there is uncountable infinity of topologically accessible \mathbf{v}_{EE} solutions with equivalent topology of \mathbf{v}_{EE} - *lines*. This concludes the proof of the Moffatt Theorem.

As a follow up I would like to make a few remarks.

1. First of all it is necessary to look a little deeper at the meaning of equivalent topology. To this end it we go back to the well known fundamentals of inviscid fluid mechanics (as was done by Moffatt in his paper of 1985a). It was briefly mentioned above that a generic flow in incompressible inviscid fluid can be seen as volume conserving diffeomorphic mappings of all the fluid elements on themselves caused by the flow generating pressure gradient. There are an infinite number of such mappings and all of them are usually interpreted as the elements of an infinite-dimensional group of volume conserving Diffeomorphisms (Arnold, 1974).

In Lagrangian reference frame (also introduced by Euler) the flow of inviscid fluid is considered by following the trajectories of all fluid elements instead of considering the flow in relation to a an observer in affixed reference frame. It is not practical for specific calculations, but allows important methodological observations that are difficult to make from the Euler equations directly. Let us introduce a vector field $\mathbf{x}(\mathbf{s}, t)$, characterizing the totality of all fluid elements trajectories in space/time. The three component parameter \mathbf{s} labels the fluid elements. While \mathbf{s} spans all allowed values the trajectories $\mathbf{x}(\mathbf{s}, t)$ fill in densely the whole fluid domain. We chose for \mathbf{s} the initial positions of fluid elements, thus the parametric equation: $\mathbf{s} = \mathbf{x}(\mathbf{s}, t = 0)$. The

trajectories $\mathbf{x}(\mathbf{s}, t)$ are related to the Euler velocity field $\mathbf{v}(\mathbf{x}, t)$ as follows:

$$\partial_t \mathbf{x}(\mathbf{s}, t) = \mathbf{v}\{\mathbf{x}(\mathbf{s}, t), t\}. \quad (\text{A.19})$$

To pass over from the Eulerian reference frame to Lagrangian means the change of coordinates so that the independent variables (\mathbf{x}, t) are substituted by (\mathbf{s}, t) , and the unknown $\mathbf{v}(\mathbf{x}, t)$ by $\mathbf{v}\{\mathbf{x}(\mathbf{s}, t), t\}$, the two related by (A.19). The Lagrangian trajectories $\mathbf{x}(\mathbf{s}, t)$ are determined for any given initial Lagrangian velocity field $\mathbf{v}_L = \partial_t \mathbf{x}(\mathbf{s}, t)|_{t=0}$. It is clear that the set of coordinates (\mathbf{s}, t) is generally non stationary, curvilinear and non orthogonal. Indeed, consider at $t = 0$ a plane, say $\mathbf{s}_1 = \text{const}$. Then at the next time moment $t > 0$ this will generally become a curved surface composed of the same fluid elements which have changed their relative positions. The change of position is caused by the pressure gradient, which is the only real force in Lagrangian representation and itself in incompressible fluids is a complicated function of the global velocity distribution velocity, while the nonlinear terms in the Euler equations went over to the Lagrangian non inertial reference frame associated forces. While the fluid elements remain at rest in Lagrangian reference frame, the coordinate system evolves, as it is effectively frozen into fluid elements. Inevitably therefore the coordinate system is non stationary, non orthogonal and curvilinear.

As was stated before kinematically a fluid motion in a compact domain can be interpreted as a diffeomorphic mapping, differentiable and invertible, by which each fluid element that is located at $\mathbf{x}(\mathbf{s}, t = 0)$, for $t > 0$ is convected to be located somewhere else at $\mathbf{x}(\mathbf{s}, t)$. Thus the total flow can be interpreted as a trajectory in an infinitely dimensional configuration space. Any deformation of an incompressible continuum corresponding to volume conserving diffeomorphic mapping, actually a reshuffling of the fluid elements, can be represented in differential form as a coordinate transformation as follows:

$$\begin{aligned} dx^i(\mathbf{s}, t) &= \partial x^i(\mathbf{s}, t) / \partial x^j(\mathbf{s}, t) dx^j(\mathbf{s}, t = 0), \\ \text{Det} \|\partial x^i(\mathbf{s}, t) / \partial x^j(\mathbf{s}, t)\| &= 1. \end{aligned} \quad (\text{A.20})$$

All these transformations together, if they are differentiable and invertible, form an infinite-dimensional group of diffeomorphisms. But generally the deformations of a continuum although smooth are not necessarily invertible. They would be of course if the class of flows

that we consider consists of analytical solutions. But the very existence and actually inevitability of the flows with tangential discontinuities in the velocity field in the Eulerian frame means that we must abandon the invertibility assumption. Finite number of deformations cannot form nonanalyticity from an initially smooth velocity field, which means as explained by Moffatt, that it forms asymptotically in the limit $t \rightarrow \infty$.

The total topology of thus evolving fields in the limit of $t \rightarrow \infty$ is not strictly the same any more. For instance the magnetic field $\lim \mathbf{B}(t \rightarrow \infty) = \mathbf{B}_M$ remain the same as $\mathbf{B}(t = 0)$ - *lines*. But the topology of the current \mathbf{j}_M - *lines* has changed by comparison with $\mathbf{j}(t = 0)$ - *lines*, as a result of the current sheets formed in \mathbf{j}_M . The same is true respectively for the velocity field and the vorticity field.

2. I would like to make a comment on the issue that I have not seen discussed in literature, although it may be understood by others as obvious. The Euler equations in both representations are of course the Hamiltonian equations, as was proved rigorously first by Arnold and since then reiterated in various ways in a number of publications. Being such they must be time reversible. However, as soon as we allow for the deformations not to be invertible in the limit $t \rightarrow \infty$, which necessary for the development of inescapable velocity field nonanalyticity in this limit, the time reversibility is abandoned. There is a very long debate that does not seem finding mathematical resolution whether the general solution of the Euler equations is regular in space/time or forms a singularity in finite time. If the latter is true then the time reversibility is not an issue. However, less stringent and definitely existing nonanalyticity of Euler flows is sufficient to break the time reversibility by itself. In real fluids described by the dissipative Navier-Stokes equations the matter resolves itself of course. But it seems that the seeds of irreversibility lie already with the nondifferentiable solutions of the Euler equations.

3. The next comment I would like to make concerns the distinctions between the Euler flows \mathbf{v}_E and \mathbf{v}_{EE} . As was pointed out by Moffatt the streamlines of \mathbf{v}_{EE} are ergodic and volume filling while the streamlines of \mathbf{v}_E are not. There is another distinction that may seem trivial but I have not met it discussed in literature.

It was pointed out above that for each \mathbf{v}_{EE} there exists analogous non-stationary solution of the Navier-Stokes equations $\mathbf{v}_{NE} =$

$\zeta\omega_{NE} = \mathbf{v}_{EE} \exp(\nu\zeta^2 t)$. Since there is an uncountable multitude of \mathbf{v}_{EE} flows there must be also an uncountable multitude of \mathbf{v}_{NE} flows as well. Since the Navier-Stokes equations do not allow surfaces of discontinuities of the velocity field it is clear that the analogy fails for these surfaces. But in total fluid domain, except the surfaces that become the seats of intensive viscous dissipation, the statement remains true. The rest is the problem of stability of \mathbf{v}_{NE} .

In this context it is important to distinguish between the stability of \mathbf{v}_{NE} flows and the stability of \mathbf{v}_{EE} flows. The latter was considered by Moffatt (1986) who concluded that \mathbf{v}_{EE} flows are absolutely unstable in difference with generally stable *dynamically accessible* \mathbf{B}_M solutions. This is easy to understand because the surfaces of tangential discontinuities in the flows of inviscid fluids are subject to absolute Helmholtz instability. This instability is widely believed to be the reason for the curdling of vorticity sheets that can only be balanced by the molecular viscosity in real fluids. However, the stability of non stationary \mathbf{v}_{NE} flows is much richer problem and the results may be non-trivial (Libin, et.al., 1987; Libin, 2008).

In difference to \mathbf{v}_{EE} all other Euler flows generally do not have at all the analogous non stationary solutions of the Navier-Stokes equations.

4. So what all of the above discussion has to do with turbulence? It was conjectured by Moffatt that since there is an uncountable multitude of Euler flows they may serve as attractors in the functional space of all the solutions of the real Navier-Stokes equations. However, except of ergodicity of \mathbf{v}_{EE} as compared with \mathbf{v}_E it was not really clear what is so special about Beltrami flows as far as the reduction on nonlinearity is concerned. This reduction can be conjectured due to \mathbf{v}_E flows. Such flows do not require generally contiguous sub-domains and in this sense the reduction can be local in space. We know experimentally that turbulence shows contiguous Beltrami sub-domains that we called **BCC**. Whether there is also a local in space attraction to \mathbf{v}_E flows we don't know. But even if there is one I doubt that this can make real dynamical impact. The reason for this, as we just discussed, is that in the Navier-Stokes equations we must rather consider the role of non stationary solutions. It is probably these solutions that we actually see as **BCC** structures and not purely inviscid \mathbf{v}_{EE} flows. The distinction is subtle but probably

important on the level of stability problem.

The functional integral representation of Sections 8 and 9 of course explicitly depends on the space/time realizations of the velocity field. The **BCC** fundamentally live in space/time and consist of Trkal flows rather than pure Beltrami flows. Still I do not suggest changing the name of **BCC** to **TCC** since we are all used to Beltrami flows terminology.

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Editorial Note

Unfortunately, after submission of the paper, the Author had a serious accident and was unable to read the proof.